

Oscillating hysteresis in the q -neighbor Ising model

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We modify the kinetic Ising model with Metropolis dynamics, allowing each spin to interact only with q spins randomly chosen from the whole system, which corresponds to the topology of a complete graph. We show that the model with $q \geq 3$ exhibits a phase transition between ferromagnetic and paramagnetic phases at temperature T^* , which linearly increases with q . Moreover, we show that for $q = 3$ the phase transition is continuous and that it is discontinuous for larger values of q . For $q > 3$, the hysteresis exhibits oscillatory behavior—expanding for even values of q and shrinking for odd values of q . Due to the mean-field-like nature of the model, we are able to derive the analytical form of transition probabilities and, therefore, calculate not only the probability density function of the order parameter but also precisely determine the hysteresis and the effective potential showing stable, unstable, and metastable steady states. Our results show that a seemingly small modification of the kinetic Ising model leads not only to the switch from a continuous to a discontinuous phase transition, but also to an unexpected oscillating behavior of the hysteresis and a puzzling phenomenon for $q = 5$, which might be taken as evidence for the so-called mixed-order phase transition.

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I. INTRODUCTION

As noted by Bar and Mukamel [1], *the classification of equilibrium phase transitions is a narrative in statistical physics and quantum field theory*. Interestingly, the Ising model has left its mark on many of the major breakthroughs in this story. Traditionally, according to Ehrenfest's idea from 1933, phase transitions were classified as n th order based on the discontinuity of the n th derivative of the thermodynamic potential [2]. Although the discussion of the validity of this classification began right after it was introduced (in the context of superfluids and superconductors), it was Onsager's analytical solution in 1944 of the two-dimensional Ising model without external field that ultimately showed the inadequacy of Ehrenfest's approach. By the 1970s, a binary classification into discontinuous (first-order) and continuous phase transitions was increasingly adopted [3].

The latter is based on the idea of an order parameter, which can be introduced for any phase transition [4,5]. A continuous transition occurs when the discontinuity in the jump of the order parameter at the transition point approaches zero. In most cases, characteristic phenomena attributed to each of the two types of transitions can be observed. For instance, continuous transitions possess universal features (which is tightly related to the divergence of a correlation length [1,6]), while discontinuous phase transitions do not. On the other hand, metastability, hysteresis, and phase coexistence are usually observed in discontinuous phase transitions and simultaneously are absent in continuous phase transitions [7].

However, it has been noticed in a number of cases—both for equilibrium and nonequilibrium systems—that the dichotomy between continuous and discontinuous transitions fails in the sense that the order parameter jumps but the correlation length diverges [1,6,8–12]. Such phase transitions are usually referred to as mixed-order phase transitions (MOTs) [1,6]. Again, the possibility of a MOT was suggested for the first time in the context of the Ising model [13]. Recently, Bar and Mukamel introduced and analyzed an exactly solvable one-dimensional Ising model with long-range interactions and rigorously showed

that it exhibits a mixed-order transition—the order parameter is discontinuous (as in first-order transitions), while the correlation length diverges (as in continuous transitions) [1,6].

The above facts not only confirm the prominent role of the Ising model in the classification of phase transitions, but also show that the Ising model, despite its “advanced age,” can still surprise. In this paper, we show that a seemingly small modification of the kinetic Ising model—in which a randomly chosen spin interacts only with its q neighbors—leads to surprising results when confronted with the common knowledge in the theory of phase transitions:

- (1) a switch from a continuous to a discontinuous phase transition at $q = 4$,
- (2) an unexpected oscillatory behavior of the hysteresis, expanding for even values of q and shrinking for odd values of q , and
- (3) a strange behavior for $q = 5$, i.e., a jump of the order parameter coinciding with lack of hysteresis (in computer simulations).

Because, as in many other kinetic models, our system is not described by the Hamiltonian, we are not able to calculate the free energy, as, e.g., in [1,6], and determine the type of transition on its basis. However, due to the mean-field-like nature of the model, we are able to derive the analytical form of transition probabilities and therefore calculate not only the probability density function (PDF) of the order parameter, but also the hysteresis and the effective potential.

II. THE MODEL

The idea to consider exactly q neighbors, no matter what the actual number of neighbors on a given graph, is borrowed from the q -voter model [14], originally proposed to introduce nonlinearity in the voter dynamics at the microscopic level. Within the q -voter model, each spin is described by a dynamical variable $S_i = \pm 1$ and interacts with a set of q neighbors. If all q neighbors share the same state, the spin conforms to this state. In the other case, the spin flips

with probability ϵ . It is worthwhile to notice here that the one-dimensional q -voter model with $q = 2$ is identical to the Ising model with generalized zero-temperature Glauber dynamics [15], in which a spin flips with probability $p = 1$ in the case of energy decrease and with probability $p = W_0$ in the case of energy conservation. If we denote $W_0 \equiv \epsilon$, the time evolution of a single spin for both models can be written as follows:

$$S'_i = \begin{cases} 1 & \text{with } p = 1 & \text{if } S_{i-1} = S_{i+1} = 1, \\ -S_i & \text{with } p = W_0 & \text{if } S_{i-1}S_{i+1} = -1, \\ -1 & \text{with } p = 1 & \text{if } S_{i-1} = S_{i+1} = -1, \end{cases} \quad (1)$$

where for brevity we use the notation $S'_i \equiv S_i(t + \Delta t)$ and $S_i \equiv S_i(t)$, and $W_0 = 1$ corresponds to the Metropolis, whereas $W_0 = 1/2$ corresponds to the original Glauber dynamics [15].

For higher dimensions, both models are not equivalent even in zero temperature and even for q equal to the number of the nearest neighbors. The q -voter model requires a unanimous state of all q neighbors to influence spin S_i , whereas for the Ising model a majority is sufficient, which follows from the Hamiltonian

$$H = - \sum_{i,j} S_i S_j. \quad (2)$$

However, one could consider the q -voter model with threshold $r = 1/2$ (i.e., majority needed to influence the spin) [16] and then again both models would be equivalent at zero temperature.

The behavior of the Ising model described by the Hamiltonian in Eq. (2) under zero-temperature Glauber dynamics is very interesting, exhibiting a slow relaxation related to a metastable state [15,17,18]. However, here we focus on another problem related to the kinetic Ising model, inspired by the analogy between the one-dimensional kinetic Ising model with zero-temperature Glauber dynamics and the q -voter model with $q = 2$. We ask the following question: *What would be the behavior of a modified kinetic Ising model—in which every spin interacts with a set of q neighbors randomly chosen from the set of all its neighbors—if we introduced a temperaturelike parameter $T > 0$?*

The algorithm of a single time step of the q -neighbor Ising model consists of three consecutive steps:

- (i) Randomly choose a spin S_i and from all its neighbors choose a subset of q neighbors, nn_q .
- (ii) Calculate the value of the following function, based on the Hamiltonian in Eq. (2), for the original state of the i th spin:

$$E(S_i) = -S_i \sum_{j \in nn_q} S_j, \quad (3)$$

and the value of the same function for the flipped i th spin, i.e., $E(-S_i)$.

- (iii) Flip the i th spin with probability $\min[1, e^{-\Delta E/T}]$, where $\Delta E = E(-S_i) - E(S_i)$.

We would like to stress here that minimizing the function given in Eq. (3) does not necessarily lead to the minimization of the whole energy of the system given by Eq. (2), in contrast to the equilibrium Ising model in which we sum interactions over all nearest neighbors. In general, the model could be considered on an arbitrary graph, but here we focus on the

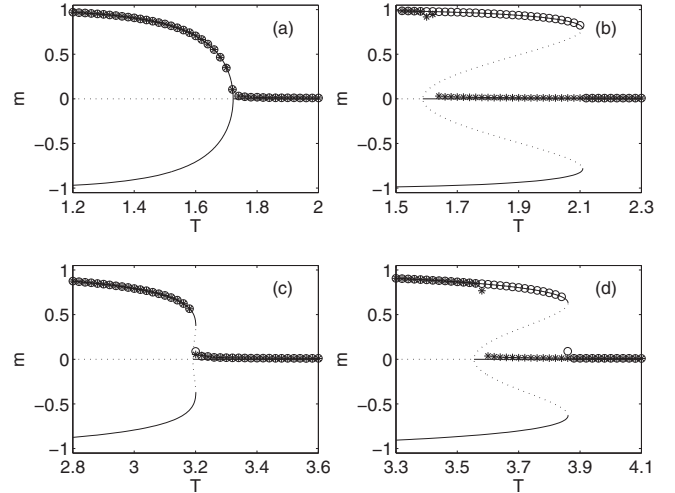


FIG. 1. Dependencies between steady values of the magnetization m and the temperaturelike parameter T for (a) $q = 3$, (b) $q = 4$, (c) $q = 5$, and (d) $q = 6$. Symbols represent results obtained from Monte Carlo simulations for a system of size $N = 10^5$ and two types of initial conditions—fully ordered (o) and disordered (*). Results were collected after 10^3 MCS (i.e., MC steps) and averaged over 10^4 samples. Solutions of Eq. (24) are presented by lines—solid for stable solutions and dotted for unstable.

complete graph, which allows for an analytical treatment. In such a case, the magnetization defined as

$$m(t) = \frac{1}{N} \sum_{i=1}^N S_i(t) \quad (4)$$

fully describes the state of the system and simultaneously, as usual, is used as the order parameter.

III. RESULTS

We investigate the model using an analytical approach and Monte Carlo (MC) simulations. It is known that to decide on the type of phase transition is quite difficult in computer simulations and in such a situation measuring the hysteresis of the order parameter is a demanding task [10]. Therefore, in the case of MC, we start from two types of initial conditions—fully ordered ($m = 1$), which corresponds to zero temperature, and completely random ($m = 0$), which corresponds to a high temperature. For each value of the temperaturelike parameter T , we measure the stationary value of the magnetization defined by Eq. (4). We have checked that averaging over time gives the same result as averaging over samples. However, when using the time average, it is easier to distinguish between continuous and discontinuous phase transitions looking solely at the order parameter as a function of T . Starting from two types of initial conditions should allow us to identify the type of the transition on the basis of the hysteresis, but, as we will see later, the hysteresis can also be misleading.

Dependencies between the stationary magnetization m and the temperaturelike parameter T are presented in Fig. 1. A phase transition between ordered and disordered phases is observed for all values of $q \geq 3$. For $q = 3$, there is no jump in the order parameter m and no hysteresis, which indicates a

continuous phase transition. For $q = 4$ and $q = 6$, the jump of the order parameter and the hysteresis indicate a discontinuous phase transition. However, in the case of $q = 5$, we are not able to identify unambiguously the type of the transition—the jump of the order parameter is observed and simultaneously there is no hysteresis, similarly as observed in [11].

However, thanks to the topology of a complete graph, which corresponds to the mean-field type of approach, we are able, following the reasoning presented in [19,20], to write the master equation that allows us to calculate the stationary probability density function of order parameter $\rho(m)$, as well as the rate equation for m in the infinite system and an effective potential V , which helps to distinguish between continuous and discontinuous phase transitions.

In a single time step Δt , three events are possible: The number of spins “up”

- (1) increases by 1 ($N_\uparrow \rightarrow N_\uparrow + 1$) and simultaneously $m \rightarrow m + 2/N$,
- (2) decreases by 1 ($N_\uparrow \rightarrow N_\uparrow - 1$) and simultaneously $m \rightarrow m - 2/N$, or
- (3) remains constant.

The first and the second changes occur with the corresponding probabilities:

$$\begin{aligned} \gamma^+ &= \sum_{k=0}^q \binom{q}{k} \frac{\prod_{j=1}^{q-k} (N_\uparrow - j + 1) \prod_{j=1}^{k+1} (N_\downarrow - j + 1)}{\prod_{j=1}^{q+1} (N - j + 1)} \\ &\quad \times \min[1, e^{\frac{2}{T}(q-2k)}], \\ \gamma^- &= \sum_{k=0}^q \binom{q}{k} \frac{\prod_{j=1}^{q-k} (N_\downarrow - j + 1) \prod_{j=1}^{k+1} (N_\uparrow - j + 1)}{\prod_{j=1}^{q+1} (N - j + 1)} \\ &\quad \times \min[1, e^{\frac{2}{T}(q-2k)}]. \end{aligned} \quad (5)$$

The time evolution of the probability density function of m is given by the master equation [21]

$$\begin{aligned} \rho(m, t + \Delta t) &= \gamma^+(m - 2/N)\rho(m - 2/N, t) \\ &\quad + \gamma^-(m + 2/N)\rho(m + 2/N, t) \\ &\quad + [1 - \gamma^+(m) - \gamma^-(m)]\rho(m, t), \end{aligned} \quad (6)$$

where $\rho(m, t)$ is the PDF of m at time t . In the case of the finite system, it is convenient to also write the master equation for N_\uparrow ,

$$\begin{aligned} \varrho(N_\uparrow, t + \Delta t) &= \gamma^+(N_\uparrow - 1)\varrho(N_\uparrow - 1, t) \\ &\quad + \gamma^-(N_\uparrow + 1)\varrho(N_\uparrow + 1, t) \\ &\quad + [1 - \gamma^+(N_\uparrow) - \gamma^-(N_\uparrow)]\varrho(N_\uparrow, t). \end{aligned} \quad (7)$$

A. Results for a finite system

Analytically solving the master equation is not an easy task, but exact formulas (5) for γ^+ and γ^- allow for a numerical solution of the equation. The most straightforward method is to iterate master equation (6) or (7). This method allows us to calculate the probability density function of m or N_\uparrow at any time and for any size of the system.

Let us compare the time evolution of PDF $\rho(m)$ obtained from master equation (6) and from Monte Carlo simulations (see Fig. 2). As usual, a single MC step (MCS) consists of N

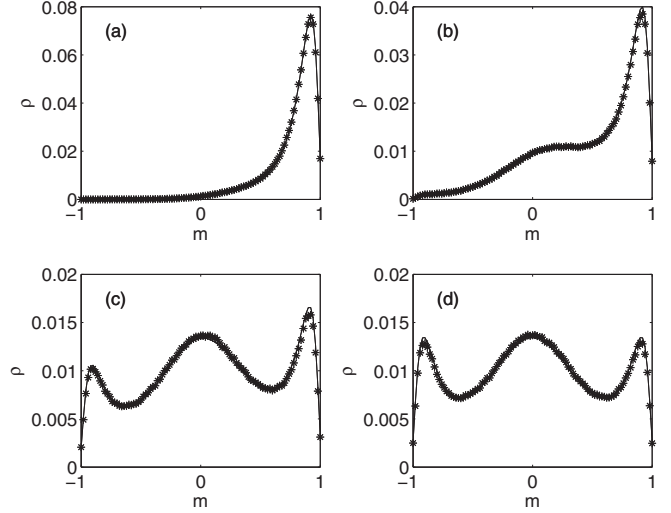


FIG. 2. The probability density function of m for the system of size $N = 100$ for $q = 4$ and $T = 2.1788$ after (a) 10, (b) 30, (c) 100, and (d) 400 MCS, which corresponds to the steady state—in this case, the coexistence of two ordered phases $m_+ = -m_- > 0$ and one disordered phase $m_0 = 0$ is seen. Monte Carlo results are presented as symbols (*) and the numerical solutions of master equation (6) are presented as the solid lines. In the case of MC simulations, the empirical PDF is based on 10^6 samples.

elementary updates, which means that $\Delta t = 1/N$. In Fig. 2, we see that even for a small system of size $N = 100$, the agreement between MC simulations and the numerical solution of master equation (6) is very satisfactory.

Eventually the system always reaches the steady state: $\rho(m, t) = \rho(m)$. If we are interested only in the stationary solution of the master equation, there is no need to iterate it, i.e., calculate the PDF in subsequent time steps, which is time consuming. We can obtain the stationary PDF directly from the condition that in the stationary state, the probability density function is time independent [17]. For convenience, let us denote N_\uparrow by K , $\varrho(N_\uparrow, t + \Delta t)$ by ϱ'_K , and $\varrho(N_\uparrow, t)$ by ϱ_K . Using the condition for the stationary state,

$$\varrho'_K = \varrho_K, \quad (8)$$

we obtain from master equation (7) the following relation:

$$\varrho_K \gamma_K^- - \varrho_{K-1} \gamma_{K-1}^+ = \varrho_{K+1} \gamma_{K+1}^- - \varrho_K \gamma_K^+. \quad (9)$$

Explicitly writing the master equation for $K = 0, 1, 2, \dots$, we check that due to the boundary condition $K \in [0, N]$ (i.e., $\gamma_0^- = \gamma_N^+ = 0$), the above relation can be replaced by the detailed balance condition,

$$\varrho_{K+1} \gamma_{K+1}^- = \varrho_K \gamma_K^+. \quad (10)$$

Therefore, the stationary PDF can be calculated directly from the relation

$$\varrho_{K+1} = \frac{\gamma_K^+}{\gamma_{K+1}^-} \varrho_K. \quad (11)$$

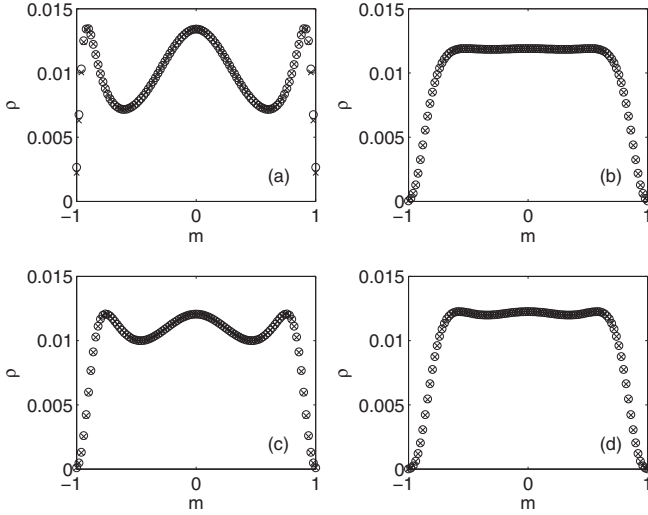


FIG. 3. The probability density function of an order parameter m at $T = T^*$ for the system of size $N = 100$ for (a) $q = 4$, (b) $q = 5$, (c) $q = 6$, and (d) $q = 7$. Results obtained from the exact relation (11) are denoted by the symbol (o) and from the general solution (16) of the Fokker-Planck equation by (x).

For an arbitrary value of ϱ_0 , we are able to calculate $\varrho_1, \varrho_2, \dots, \varrho_N$ and then, normalizing

$$\varrho_K \rightarrow \frac{\varrho_K}{\sum_{i=1}^N \varrho_i}, \quad (12)$$

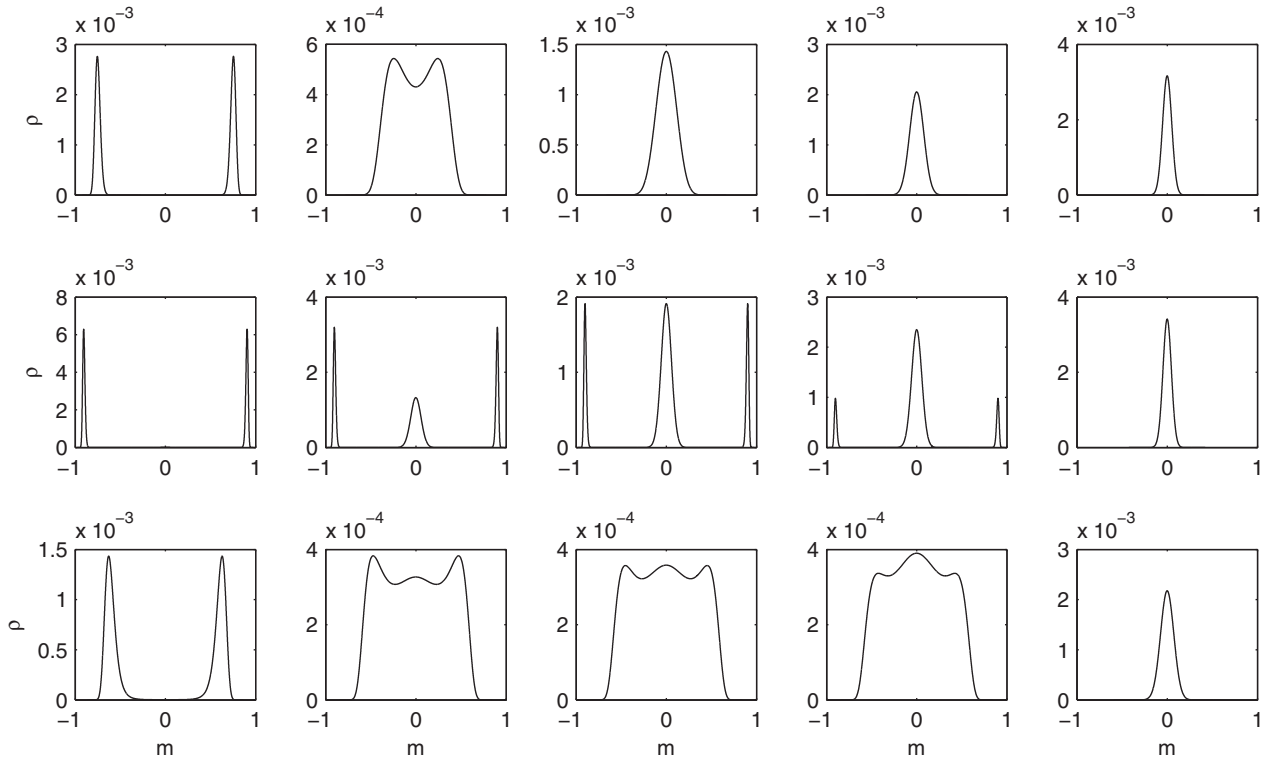


FIG. 4. The stationary probability density function of order parameter m for the system of size $N = 5000$. Different panels correspond to different values of q and T : q increases from top to bottom, whereas T increases from left to right. Specifically, results for $q = 3$ are presented in the top row, for $q = 4$ in the middle row, and for $q = 5$ in the bottom row. In the left column, $T \ll T^*$, i.e., far below the phase transition, is presented. In the next column, T is slightly below the transition point. The middle column corresponds to the transition point, i.e., $T = T^*$. In the next column, T is slightly above the transition point. Finally, in the right column, $T \gg T^*$, i.e., far above the phase transition, is presented.

we find the PDF of the number of spins “up.” A similar idea has been recently used in the case of the q -voter model with inflexible zealots [22].

For large, but finite systems ($1 \ll N < \infty$), we can also solve master equation (6) analytically, approximating it by the continuous Fokker-Planck equation, following the reasoning from [23]:

$$\frac{1}{N} \frac{\partial}{\partial t} \rho(m, t) = \frac{4}{N^2} \frac{\partial^2 D \rho(m, t)}{\partial m^2} - \frac{2}{N} \frac{\partial F \rho(m, t)}{\partial m}, \quad (13)$$

where

$$D = \frac{\gamma^+(m) + \gamma^-(m)}{2} \quad (14)$$

is the diffusion coefficient and

$$F = \gamma^+(m) - \gamma^-(m) \quad (15)$$

is the drift, which simultaneously can be treated as an effective force that drives m up or down [20,23].

In [23], it has been shown that Eq. (13) has the following stationary solution:

$$\rho(m) = \frac{C}{D} \exp \int \frac{NF}{2D} dm, \quad (16)$$

where C is a normalizing constant, such that

$$\int_{-1}^1 \rho(m) dm = 1. \quad (17)$$

Although the above solution has been derived under the assumption that the system size is large, i.e., $N \gg 1$, which allows for a continuous approximation of discrete equation (6), the obtained results agree very well with exact results (11) already obtained for $N = 100$ (see Fig. 3). Therefore, from now on, we use formula (16) to calculate the stationary value of the PDF for finite systems.

Several examples of $\rho(m)$ for the system of size $N = 5000$ are shown in Fig. 4:

(i) It is seen that for $q = 3$ (top row in Fig. 4), there are two maxima that represent two symmetrical ordered phases $m_+ = -m_- > 0$ below the critical point $T = T_c$. With increasing $T < T_c$, the maxima are approaching each other and eventually form a single maximum at $T = T_c$, which is typical for a continuous phase transition.

(ii) A different situation is seen for $q = 4$ (middle row in Fig. 4). Again, for small values of T , there are two peaks, but with increasing T , they almost do not change their positions. Instead, for $T = T_1$, a third peak appears. The central peak is initially lower than the remaining two maxima, which means that it represents a metastable state. As $T > T_1$ increases, the third peak grows and, for $T = T^*$, all three maxima have the same height. For $T > T^*$, the central maximum dominates over the other two, which means that state $m = 0$ is stable and the remaining two are metastable. Finally, for $T = T_2$, only the center peak remains. This is a typical picture for a discontinuous phase transition, which takes place at $T = T^*$.

(iii) For $q = 5$ (bottom row in Fig. 4), for which no hysteresis was seen in the computer simulations, the situation is slightly different than for $q = 4$, but the only difference between these two cases involves the separation between the peaks. Around the transition point $T = T^*$ (i.e., for $T_1 < T < T_2$), again three local maxima are observed, but they are not separated as for $q = 4$. Instead, they create a crowning of a single wide hump. In the next section, we show that for $N \rightarrow \infty$, the distance between the peaks does not vanish, which suggests that the transition is discontinuous [24]. Moreover, we show that the hysteresis for $q = 5$ exists, but is too small to be visible in MC simulations.

Although to obtain the stationary PDF $\rho(m)$ we do not have to iterate master equation (7), it is instructive to look at the relaxation time τ as a function of the system size N . We start from the initial condition with all spins up, which means that $\rho(m,0) = \delta(m-1)$, we iterate master equation (7), and we measure the time τ until the stationary PDF $\rho(m)$ is reached. From Fig. 4, we expect that the characteristic time τ will be larger for even values of q because the separation between peaks of $\rho(m)$ is significantly larger in this case. As expected, for even values of q , the relaxation time τ grows rapidly with the system size N and does not scale with N (see Fig. 5). For odd values of q , the relaxation time τ grows with N much slower, and for $q = 3$ and $q = 5$, power laws are observed. It is worthwhile to comment on the result for $q = 5$ in the context of both a directed percolation model and a nonequilibrium kinetic Ising model (NEKIM) [7,25,26]. In both models, to determine the dynamical exponent z , one usually starts with a fully occupied lattice. In such a case, at the critical point, the spatial correlation length grows in time as

$$\xi(t) \sim t^{1/z}. \quad (18)$$

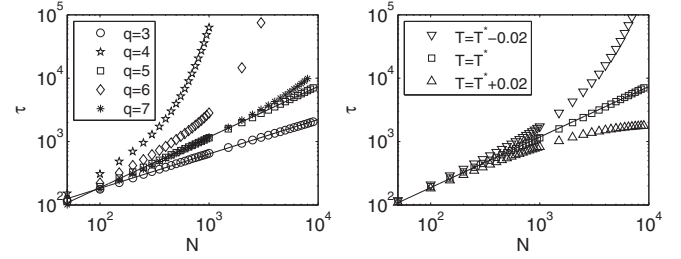


FIG. 5. The relaxation time from the initial state $\rho(m,0) = \delta(m-1)$ to the stationary $\rho(m)$ given by Eq. (16) as a function of system size N in the log-log scale. Solid lines represent the power-law fits $\tau \sim N^{B(q)}$. In the left panel, results for $q = 3, 4, 5, 6, 7$ at $T = T^*$ are shown. The right panel presents results for $q = 5$ at $T = T^*$, as well as slightly below and above the transition point.

On the other hand, it is known that in finite systems, the characteristic time τ to reach the absorbing states scales with the lateral size L as

$$\tau \sim L^z = N^{z/d}, \quad (19)$$

where d is the spatial dimension, i.e., $N = L^d$. Our model is investigated on the complete graph and therefore we cannot directly measure the spatial correlation length ξ . However, as discussed in the previous paragraph, we can measure the characteristic time τ as a function of N , which indirectly gives us the information about the correlation length. Results presented in Fig. 5 suggest that for $q = 3$ and $q = 5$, the relaxation time indeed scales with the system size, which is expected for the continuous phase transitions. The latter result reopens the discussion of the type of phase transition for $q = 5$. Let us summarize that so far we have observed, for $q = 5$,

- (1) a jump of the order parameter,
- (2) a lack of hysteresis,
- (3) an almost flat $\rho(m)$, what can be treated as an indicator of MOT [8,13], and
- (4) finite-size scaling of τ , which suggests that the correlation length diverges.

Despite all of these results, we are cautious with the ultimate statement that there is a mixed-order phase transition for $q = 5$. Our caution stems mainly from the fact that our results are only semianalytical. Although the master equation (7) has an analytical form, all characteristics, including the relaxation time τ , have been derived numerically. Moreover, the hysteresis for $q = 5$ is nearly zero, but not exactly zero, which will also be discussed in the next section. Similarly, $\rho(m)$ is almost flat, but not entirely flat. We would like to note that for $q = 7$, the deviation from the power law (see the left panel in Fig. 5) is very small, similar to the hysteresis (see the next section). Furthermore, analogous to $q = 5$, around the transition point $T = T^*$, the PDF of m consists of three local maxima that create a crowning of a single wide hump. A similar behavior is observed for all odd values of $q > 3$, only (i) the deviation from the power law, (ii) the hysteresis, (iii) the separation between the peaks of $\rho(m)$, and (iv) the jump of the order parameter all grow with q . Therefore, the systematic behavior of the system for odd values of q leads us to believe that the phase transition is discontinuous also for $q = 5$. However, we are ultimately unable to solve this puzzle,

and certainly additional studies, including other topologies, are needed.

B. Results for an infinite system

The results obtained from formula (16) for the finite system, which agree with the exact results obtained from (11), suggest that there is a continuous phase transition for $q = 3$ and discontinuous phase transitions for $q \geq 4$. In typical discontinuous transitions, the PDF of order parameter m is double humped with a deepening valley between the two peaks related to each of two phases [24]. Moreover, the distance between the peaks tends to a positive constant which is equal to the jump in m . However, in finite systems, this is not enough to confirm that the transition is discontinuous. For that, one must also show that the distance between the peaks does not vanish for $N \rightarrow \infty$ [24].

In our case, we have three peaks due to the definition of order parameter m [see Eq. (4)], which distinguish between two ordered phases, $m_+ = -m_-$. To confirm that the transition is discontinuous, we have to show that the distance

$$\Delta m_{\pm} \equiv m_{\pm} - m_0, \quad (20)$$

where $m_{\pm} = |m_+| = |m_-|$, does not vanish for $N \rightarrow \infty$. We will show that indeed distance Δm_{\pm} tends to a positive constant which is equal to the jump in m for the infinite system. Moreover, for $N \rightarrow \infty$, we will show that the hysteresis exists for any $q \geq 4$ but exhibits an unexpected oscillatory behavior, expanding for even values of q and shrinking for odd values of q .

Let us denote the concentration of spins up by

$$c(t) = \frac{N_{\uparrow}}{N} = \frac{m + 1}{2}. \quad (21)$$

For the infinite system $N \rightarrow \infty$, transition probabilities (5) can be rewritten in the following compact form:

$$\begin{aligned} \gamma^+ &= \sum_{k=0}^q \binom{q}{k} c^{q-k} (1-c)^{k+1} \min[1, e^{\frac{2}{r}(q-2k)}], \\ \gamma^- &= \sum_{k=0}^q \binom{q}{k} (1-c)^{q-k} c^{k+1} \min[1, e^{\frac{2}{r}(q-2k)}], \end{aligned} \quad (22)$$

and the dynamics is described by the rate equation [17]

$$c(t + \Delta t) = c(t) + \frac{1}{N} [\gamma^+(c) - \gamma^-(c)]. \quad (23)$$

In the stationary state, $c(t + \Delta t) = c(t)$, which is equivalent to the condition that the effective force

$$F(c) = \gamma^+(c) - \gamma^-(c) = 0. \quad (24)$$

Analytically solving Eq. (24) for an arbitrary value of q is impossible, but we can easily do it numerically. The results, after converting to m using relation (21), are denoted by lines in Fig. 1 and agree very well with MC results. Let us now compare the solution of Eq. (24) with the distance between peaks for several systems with different sizes N . It is seen in Fig. 6 that Δm_{\pm} coincides with m obtained from Eq. (24), and therefore the distance between peaks m_{\pm} and m_0 at the transition point $T = T^*$ is equal to the jump of m . Moreover,

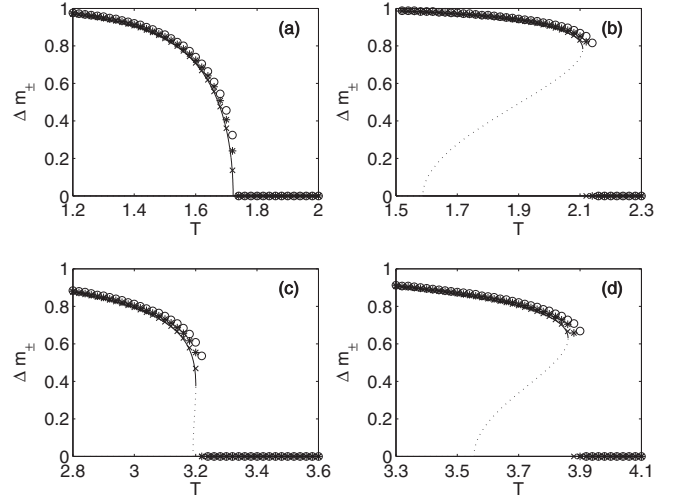


FIG. 6. Distance Δm_{\pm} between peaks m_{\pm} and m_0 of PDF $\rho(m)$ given by Eq. (16) for three system sizes: $N = 500$ (\circ), $N = 1000$ ($*$), and $N = 5000$ (\times). Lines represent stable (solid) and unstable (dotted) solutions of Eq. (24), which gives the value of m for the infinite system. It is seen that Δm_{\pm} coincides with m obtained from Eq. (24).

we see that Δm_{\pm} do not vanish for $T \leq T^*$ for any value of N , which confirms that the transition is discontinuous.

Having the effective force $F(c)$ for the infinite system, we can also calculate the effective potential,

$$V(c) = - \int F(c) dc, \quad (25)$$

which, as seen in Fig. 7, allows one to distinguish between stable (minima of the potential) and unstable (maxima of

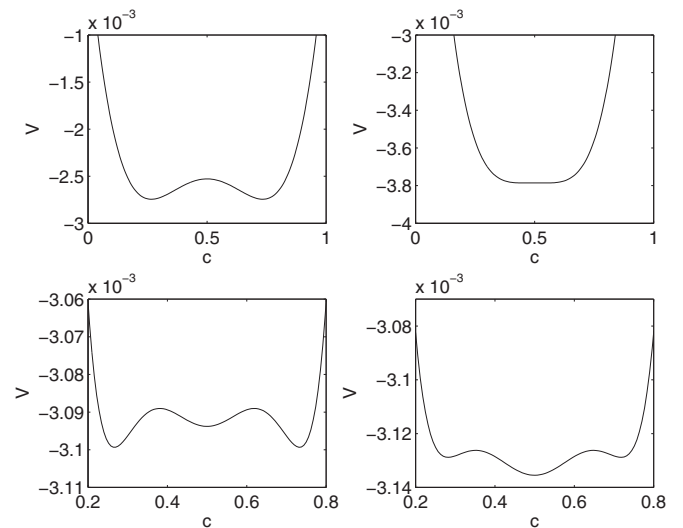


FIG. 7. Potentials given by Eq. (25) for $q = 3$ (top panels) and $q = 5$ (bottom panels). Left panels represent potentials just below the transition point, and right panels just above the transition point. It is seen that for $q = 3$, there are two ordered phases below the transition point and a single disordered phase above the transition point, which is typical for continuous phase transitions. For $q = 5$, the phase coexistence and metastable states are seen, which indicates a discontinuous phase transition.

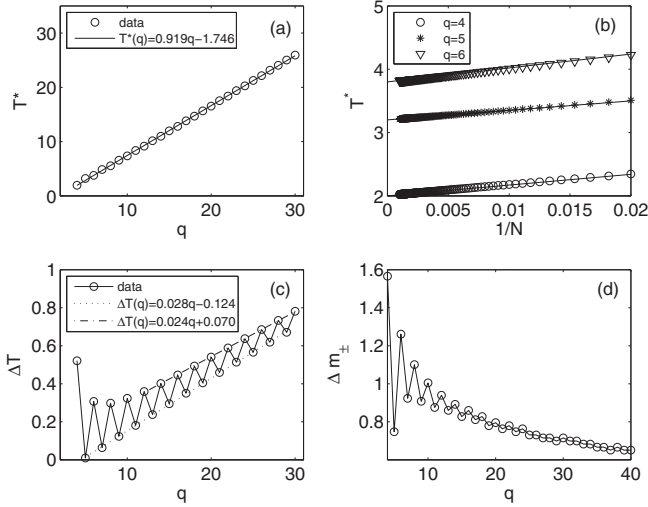


FIG. 8. (a) Transition value T^* , derived from the potential for the infinite system, can be approximated by a linear function of q . (b) Dependence between transition value $T^*(N)$ and the inverse system size, obtained from the PDF for the finite system. The results nicely scale with the system size, and $T^*(N \rightarrow \infty)$ agrees with the value obtained from the potential for the infinite system. (c) Dependence between the width of the hysteresis (i.e., $\Delta T = T_2 - T_1$) and number of neighbors q shows oscillatory behavior. (d) The jump of the order parameter also oscillates with q .

the potential) solutions of Eq. (24). Furthermore, it allows one, independently of the behavior of the PDF, to distinguish between continuous and discontinuous phase transitions, in the latter case showing phase coexistence and metastable states. Finally, it allows one to determine the transition point, which in the case of a discontinuous phase transition coincides with the value of T , for which minima corresponding to disordered and ordered phases are equal.

Having the effective force (24) and the effective potential (25), we can numerically calculate not only the stationary value of the order parameter $m(T)$ for an arbitrary value of q , but also determine the transition temperature T^* , the jump of the order parameter at $T = T^*$ as a function of q , and the width of the hysteresis defined here as the distance between the spinodal lines (see Fig. 8). More precisely, the latter can be calculated from the potential (25). For low values of T , there are two minima that correspond to ordered phases. Then, at $T = T_1$, the third minimum appears; it corresponds to the disordered phase but is shallower than other two, i.e., the disordered state is metastable. At $T = T^*$, all three minima are equal, which corresponds to the transition point. Above this value, the middle minimum corresponding to the disordered phase is the deepest and the other two represent metastable ordered states. Finally, above $T = T_2$, there is only one minimum—the disordered state is the only possible state of the system. The distance between the spinodal lines, $\Delta T = T_2 - T_1$, determines the width of the hysteresis and is presented in the top left panel of Fig. 8.

The transition point $T = T^*$ could also be derived from the PDFs for finite systems using finite-size scaling. As previously written and shown, PDF $\rho(m)$ for the system of size N has three equally high peaks at $T = T^*(N)$. The real transition

point is given by $T^* = T^*(N \rightarrow \infty)$ and it coincides with the transition point derived from the potential (see the top panels in Fig. 8).

In Fig. 8, it can be seen that transition temperature T^* increases linearly with q , whereas hysteresis width ΔT and the jump of order parameter Δm_{\pm} exhibit an oscillatory behavior, expanding for even values of q and shrinking for odd values of q . Interestingly, for $q = 5$, the width of the hysteresis is nearly zero [$\Delta T(q = 5) = 0.0091$] and, even for $q = 7$, it is still very small [$\Delta T(q = 7) = 0.0642$]. Therefore, it is difficult to see any hysteresis for $q = 5$ or even $q = 7$ in computer simulations. This fact has initially led us to the wrong conclusion of oscillatory switching from continuous to discontinuous phase transitions. However, deriving $\Delta T(q)$ from the potential confirms that it is larger than zero for all $q \geq 4$, and therefore the transition is certainly not continuous. Additionally, taking into account the behavior of PDF $\rho(m)$, as discussed above, we conclude that there is a discontinuous phase transition for any $q \geq 4$ but, interestingly, its “abruptness” ($\Delta T, \Delta m_{\pm}$) changes in an oscillatory manner with q .

IV. CONCLUSIONS

It is believed that the hysteresis is the main indicator of discontinuous phase transitions [7,10]. However, as we have shown here, the behavior of the hysteresis might be quite unexpected.

In this paper, we have introduced a seemingly small modification of the kinetic Ising model on a complete graph, allowing each spin to interact only with q neighbors. Surprisingly, this modification leads to a switch from a continuous to a discontinuous phase transition for $q = 4$ and an unexpected oscillating behavior of the hysteresis for $q \geq 4$ —it expands for even values of q and shrinks for odd values of q .

It is known that by increasing the number of interacting neighbors, the fluctuations are diminished and the transitions become sharper [25]. In equilibrium statistical mechanics, it is common that systems that exhibit a discontinuous phase transition in high space dimensions may display a continuous transition below a certain upper critical dimension [7]. We expect that at the same time, the hysteresis monotonically decays, reaching zero at the upper critical dimension. The behavior of our model is different. Although there is a switch from a discontinuous to a continuous phase transition for decreasing q , the behavior for $q \geq 4$ is very surprising. If only odd values of q were considered, then indeed we would observe a monotonic, linear increase of hysteresis ΔT (see Fig. 8). For even values of q , the dependence between ΔT and q is nonmonotonic, having the smallest value for $q = 8$. If we consider all values of q , the relation $\Delta T(q)$ is even more complex, with oscillatory behavior.

Another interesting feature of the model is the size of the hysteresis for $q = 5$: $\Delta T(q = 5) = 0.0091$, i.e., it is nearly zero, especially when compared with the transition temperature of $T^* = 3.1974$. If the model was not solvable analytically, Monte Carlo results could drive us to the wrong conclusion of a hybrid phase transition in which the jump of the order parameter coincides with no hysteresis. In fact, even finding the hysteresis does not resolve the problem of MOT because some mixed-order phase transitions may also show

hysteresis [1,6]. Because, as in many other kinetic models, our system is not described by the Hamiltonian, we are not able to calculate the free energy, as, e.g., in [1,6], and determine the type of transition on its basis. Moreover, definition (25) of the effective potential V is valid only for the infinite system. Therefore, we cannot rigorously draw a conclusion on the type of phase transition while studying the size dependence of the barrier in the potential V at the transition point. Instead, we are able to analytically calculate the probability density function of the order parameter and show that for $q \geq 4$, the behavior of the PDF is typical for discontinuous phase transitions. Even for $q = 5$, three peaks are observed, although much less separated than for $q = 4$. Because simultaneously the jump of the order parameter for $q = 5$ is very small (see Fig. 8), we conclude that this is probably the case of a weakly discontinuous transition instead of MOT. However, finite-size scaling of the relaxation time puts the latter into question and therefore we refrain from formulating an ultimate statement.

Despite the fact that most of the results presented here were derived from the master equation and not only from

MC simulations, the intuitive understanding of the nonmonotonicity of hysteresis $\Delta T(q)$ is still missing. Perhaps studying the behavior of the model on different topologies, where spatial correlations are relevant, could be helpful. Another idea would be to consider the model with annealed (fixed) connections with q neighbors instead of quenched as here. Such an approach would complicate the analytical treatment, but could potentially shed some light on the problem. There are several other directions of further studies and some preliminary results have already been obtained. However, they have not yet brought us closer to understanding this simple, yet surprising Ising-based model. Therefore, certainly the model deserves further study.

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