Phase transitions in the q-voter model with noise on a duplex clique

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We study a nonlinear q-voter model with stochastic noise, interpreted in the social context as independence, on a duplex network. To study the role of the multilevelness in this model we propose three methods of transferring the model from a mono- to a multiplex network. They take into account two criteria: one related to the status of independence (LOCAL vs GLOBAL) and one related to peer pressure (AND vs OR). In order to examine the influence of the presence of more than one level in the social network, we perform simulations on a particularly simple multiplex: a duplex clique, which consists of two fully overlapped complete graphs (cliques). Solving numerically the rate equation and simultaneously conducting Monte Carlo simulations, we provide evidence that even a simple rearrangement into a duplex topology may lead to significant changes in the observed behavior. However, qualitative changes in the phase transitions can be observed for only one of the considered rules: LOCAL&AND. For this rule the phase transition becomes discontinuous for $q = 5$, whereas for a monoplex such behavior is observed for $q = 6$. Interestingly, only this rule admits construction of realistic variants of the model, in line with recent social experiments.

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I. INTRODUCTION

Opinion dynamics is one of the most investigated sub-fields of sociophysics [1–4]. Subjectively, there are at least two important reasons why physicists study this topic. The first motivation comes from social sciences and can be described as a temptation to build a bridge between the micro and macro levels in describing social systems. Traditionally, there are two main disciplines that study social behavior: sociology and social psychology. Although the subject of the study is the same for both disciplines, the approaches usually taken are very different. Sociologists study social systems from the level of the social group whereas social psychologists concentrate on the level of the individual [5]. From the physicist's point of view this is similar to the relationship between thermodynamics and statistical physics. This analogy raises the challenge to describe and understand the collective behavior of social systems (sociology) from the level of interpersonal interactions (social psychology). The second motivation to deal with opinion dynamics is related to the development of nonequilibrium statistical physics. Models of opinion dynamics are often very interesting from the theoretical point of view [6]. One of the best examples is the famous voter model or, the recently introduced, nonlinear q-voter model [7]. Both models are based on dichotomous opinions and belong to a wide class of binary-state dynamics [1,8–10]. It should be stressed here that binary opinions are natural from the social point of view, since the dichotomous response format with 1 (yes, true, agree) and 0 (no, false, disagree) as response options is one of the most common in social experiments [11,12].

Among many other binary opinion models [2–4], the q-voter model is not only interesting from theoretical point of view, but also justified from the social point of view. In short, within the q-voter model each individual interacts with a set of q neighbors (a q-lobby) and if all q neighbors share the same state (i.e., the q-lobby is unanimous), the individual conforms to this state. As originally proposed, in the other case (disagreement) the individual changes its state with probability ϵ [7]. However, in some later publications the model with $\epsilon = 0$ was studied, as a natural generalization of the Sznajd model [13–15]. The unanimity rule can be justified based on social experiments. It has been observed in number of experiments that a small unanimous group may be more efficient than a much larger group with a nonunanimous majority [5]. In a classical series of experiments on conformity, Asch [16] found that the presence of a social supporter reduced conformity dramatically: participants of the experiment were far more independent when they were opposed by a seven person majority and had a partner sharing the same opinion than when they were opposed by a three-person majority and did not have a partner. Influence of a consistent minority on the responses of a majority has been reported by Moscovici *et al.* [17]. Also, recent neurological experiments suggest that unanimous opinions may be critical for normative influence [18].

From the physicist's point of view the q-voter model is interesting because of the rich behavior related to phase transitions [7,13,19] as well as the controversy related to the exit probability of the model [14,15]. In this paper we will focus on phase transitions driven by stochastic noise which—in the social context—may be interpreted as independence [13,20]. In social psychology, independence is recognized as one of the two types of nonconformity and means resisting influence [21]. It has been noticed that independence plays a role similar to the temperature and introduces order-disorder phase transitions [20]. Interestingly, it has been shown that, in the case of a complete graph, the phase transition changes its type from continuous to discontinuous for $q \geqslant 6$ [13,20].

Until now, the q-voter model has been studied on monoplex networks, i.e., networks that consist of only one level. However, as noted recently, interactions among individuals can be of qualitatively different nature and therefore modeled by multilevel networks [22]. In the last two years, a lot of attention has been devoted to the analysis of various dynamics on multiplex networks, including diffusion processes [23], epidemic spreading [24–26], and voter dynamics [27]. Brummitt *et al.* [28] have generalized the threshold cascade model on complex networks [29]. In [30] they further expanded the model and introduced the idea of OR and AND nodes. An OR node is activated as soon as a sufficiently large fraction of its neighbors are active in at least one level. An AND node is activated only if in each and every layer a sufficiently large fraction of its neighbors are active. We will use a related notion of OR and AND types of social influence.

Without doubt most real-world social networks consist of many levels. For instance, a student may belong to a network of classmates, a network of sport-club teammates, and a network of Facebook friends. The question is if this multilevelness is important for the macroscopic (or global) properties of the social system, such as public opinion, or not. The general answer to this important question is beyond the scope of this paper. Instead we try to answer the question within the q-voter model that takes into account two types of response to social influence: conformity and independence.

The remainder of this paper is organized as follows. In Sec. II, we briefly recall the q-voter model on a single monoplex clique. We extend this framework to multiplex cliques in Sec. III and propose three rules (GLOBAL&AND, GLOBAL&OR, and LOCAL&AND). In Sec. IV we derive rate equations that describe the time evolution of the system, for each of the three rules. Furthermore in Sec. V, based on these equations, we derive phase diagrams and compare results obtained from analytical equations with those obtained from Monte Carlo simulations. The next section is devoted to a deeper analysis of the phase transitions. In Secs. VII and VIII we go beyond the duplex clique to check to what extent our results are universal. Finally, in Sec. IX we wrap up the results and conclude.

II. THE q-VOTER MODEL ON A SINGLE MONOPLEX CLIQUE

In [13] we introduced the q-voter model with two types of stochasticity that, using the language of social sciences, could be interpreted as two kinds of nonconformity, namely anticonformity and independence [21]. In [20] we equated anticonformity with antiferromagnetic interactions, since in both cases a voter takes an opposite state to the majority. According to social psychologists, anticonformity is similar to conformity in the sense that in both cases an individual takes cognizance of the group norm; in the case of conformity agreeing with the influence group and in the case of anticonformity disagreeing. In contrast, in the case of independence the situation is not based on the opinions of others; in [20] we equated this type of nonconformity with temperature. In this case a voter takes a random state, independently of the group of influence.

Very recently another variant of independence—inflexible zealots—has been studied within the q-voter model [31]. In this version of the model, a susceptible voter adopts the opinion of a neighbor if this neighbor belongs to a group of q neighbors all in the same state, whereas inflexible zealots never change their opinion. There are two main differences between the approaches proposed in [13] and [31]. First of all, in [13] independence allows a voter to change a state with probability $\frac{1}{2}$, whereas in [31] independence means the absence of any changes. Both approaches are special cases of a more general idea of independence, introduced originally for the Sznajd model in [32], where in the case of independence a voter could change its state with probability f. This general idea of independence has been recently used in a three-state kinetic exchange opinion model [33]. The second difference between the approaches proposed in [13] and [31] is related to the *person vs situation* response to social influence [34]. In [31] a person-oriented approach has been used; i.e., each agent has been permanently assigned to one of two types: susceptible (conformist) or inflexible zealot (independent). In contrast, in [13] the situation-oriented approach has been used; i.e., the same agent could be independent in a given time step (with a certain probability) and behave like a conformist in the next moment.

Although the situation-oriented approach may seem unrealistic for some readers, surprisingly it is far more realistic than the person-oriented approach for many social processes; as has been shown in numerous experiments, a situation can almost completely prevail over personality [5]. On the other hand, personality psychologists argue that personality attributes not only exist but also shape how individuals adapt to the challenges of life [35]. In fact, there has been a longstanding and vigorous discussion on this topic among psychologists, known as the *person-situation debate*. From the empirical (experimental) point of view the situation-oriented approach is much better motivated; however, from the theoretical point of view both approaches are interesting; see [34] for a discussion in the context of agent-based modeling.

It turns out that the q-voter model with situation-oriented independence is not only better motivated, but also more interesting from a modeling point of view as it exhibits a regime-switching (or phase transition-type) behavior. Therefore it is also used in this paper. Before we proceed with describing the model on a duplex clique, let us briefly recall its formulation on a monoplex complete graph. In this case we consider a set of N individuals, which are described by the binary variables $S_i = \pm 1$ (spins "up" or "down"). At each elementary time step t we randomly choose an ith node (i.e., a voter) and a q-lobby, which is a randomly picked group of q individuals. Only if the q-lobby is self-consistent can it influence the voter. With probability $1-p$ the q-lobby (if it is homogeneous) acts on the state of the voter, which means that the voter changes state to the state of the q-lobby. With probability p the voter behaves independently: with equal probabilities flips to the opposite direction $S_i(t+1) = -S_i(t)$ or keeps its original state $S_i(t+1) = S_i(t)$. Therefore only the following changes are possible:

$$\begin{aligned}
&\underbrace{\uparrow\cdots\uparrow}_{q}\Downarrow \xrightarrow{1-p} \underbrace{\uparrow\cdots\uparrow}_{q}\Uparrow,\\
&\underbrace{\downarrow\cdots\downarrow}_{q}\Uparrow \xrightarrow{1-p} \underbrace{\downarrow\cdots\downarrow}_{q}\Downarrow,\\
&\underbrace{\cdots}_{q}\Downarrow \xrightarrow{p/2} \underbrace{\cdots}_{q}\Uparrow,\\
&\underbrace{\cdots}_{q}\Uparrow \xrightarrow{p/2} \underbrace{\cdots}_{q}\Downarrow,
\end{aligned} \tag{1}$$

where a single-line arrow represents the state of a node belonging to the q-lobby, while the state of the voter is marked with a double-line arrow. Lines 1 and 2 in (1) correspond to conformity, which occurs with probability $(1-p)$ and has the ability to change the state of the voter only when all spins in the q-lobby (single-line arrows) are in the same state. Lines 3 and 4 in (1) describe independence, which occurs with probability p. In such a case the voter behaves randomly, i.e., independently of the q-lobby flips with probability $\frac{1}{2}$ to the opposite direction or does not change with the same probability. Cases where the state of the voter does not change are not shown.

It has been shown that the system, described by the q-voter model with independence, undergoes the phase transition at $p = p_c(q)$. For $p < p_c$ the majority coexists with the minority opinion (ordered state) and for $p > p_c$ there is a status quo (disordered state) [13]. Interestingly, it occurrs that for $q \leqslant 5$ the phase transition is continuous, whereas for $q > 5$ it becomes discontinuous.

The most natural quantity that describes the macroscopic behavior of such a system is magnetization, which from the social point of view represents so called public opinion:

$$m(t) = \frac{1}{N}\sum_{i=1}^{N} S_i(t). \tag{2}$$

Moreover, in the case of a complete graph, the magnetization fully describes the state of system. In this paper we will calculate it in two ways: by Monte Carlo simulations of the microscopic system of the size N and by numerical solution of the equation describing the time evolution of the average magnetization. In the case of the Monte Carlo simulations we will calculate the ensemble average of the magnetization in the stationary state:

$$\langle m\rangle = \frac{1}{M}\sum_{j=1}^{M} m_i, \tag{3}$$

where m_i denotes the stationary value of the magnetization in the ith realization (sample) and $i = 1, \ldots, M$. In this paper we average all Monte Carlo results over $M = 10^3$ samples.

III. THE q-VOTER MODEL ON A DUPLEX CLIQUE

Let us start by defining a duplex clique, which is a particular case of a multiplex. Specific definitions of multiplex networks have been introduced in [22,36,37]. Such systems consist of distinct levels (layers), and the interconnections between levels are only between a node and its counterpart in the other layer (i.e., the same node). Here we consider a duplex clique (see Fig. 1), i.e., a network that consists of two distinct levels (layers), each of which is represented by a complete graph (i.e., a clique) of size N. Levels represent two different communities (e.g., Facebook and school class), but are composed of exactly the same people: each node possesses a counterpart node in the second level. Such an assumption reflects the fact that we consider fully overlapping levels, which is an idealistic scenario. We also assume that each node possesses the same state on each level, which means that the society consists of nonhypocritical individuals only.

It is worth stressing the difference between a duplex clique and two inter-onnected monoplex cliques (see Fig. 2), where the state of a node on one level is not directly related to the state of a node in the second layer. In interconnected monoplex cliques the interclique links fulfill the same role as intraclique

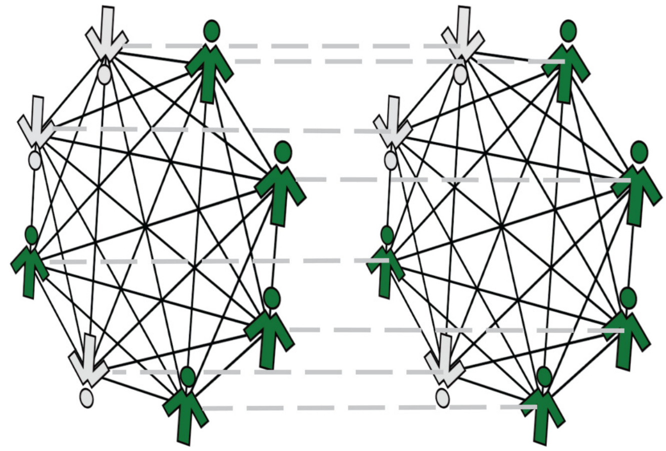

FIG. 1. (Color online) The topology of a duplex clique, i.e., a network that consists of two distinct levels (layers), each of which is represented by a complete graph (i.e., a clique) of size N. Levels represent two different communities but are composed of exactly the same people: each node possesses a counterpart node in the second level. Interconnections between levels (denoted by the gray dashed lines) are realized exclusively by connecting the node with its own counterpart (i.e., the same node) on the other level. Links within a single level (black solid lines) represent some kind of social relation (e.g., friendship).

edges. Classical voter models were analyzed on interconnected cliques in [38,39].

In this paper we investigate the q-voter model on a duplex clique. As in the case of the monoplex, we consider a set of N individuals described by the binary variables $S_i = \pm 1$, which are the same on both levels. On each level all individuals are connected with each other and therefore create a duplex clique, as shown in Fig. 1.

We consider two criteria of level dependence: one related to the status of independence (GLOBAL vs LOCAL) and one related to peer pressure (AND vs OR) (see also Table I):

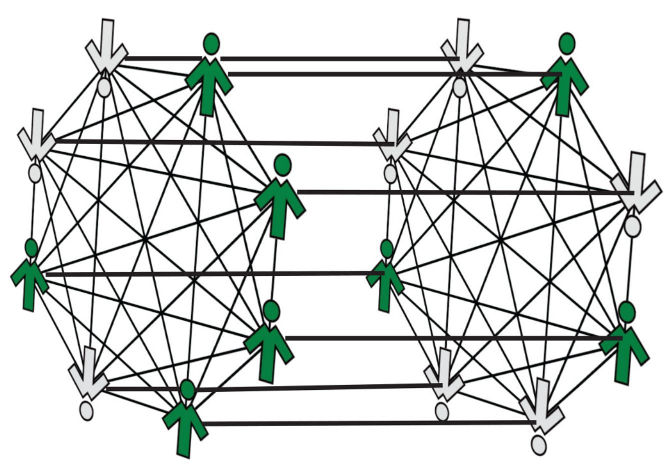

FIG. 2. (Color online) The topology of two interconnected monoplex cliques. This network consists of two levels, each of which being represented by a complete graph of size N. Levels represent two different communities and are composed of different people. Links within a single level are equal to the connections between levels (black solid lines) and represent some kind of social relation (e.g., friendship).

TABLE I. Three versions of dynamics on the multiplex clique. We do not consider the `LOCAL&OR` rule because of the difficulty of such a concept related to social unreality and algorithmic ambiguity.

	`AND`	`OR`
Global independence	(i) `GLOBAL&AND`	(ii) `GLOBAL&OR`
Local independence	(iii) `LOCAL&AND`	

(1) Criteria related to the status of independence: the `GLOBAL` rule means that an agent is independent on both levels, but the `LOCAL` rule admits a situation where a person is independent in one clique but not in the other.

(2) Criteria related to the peer pressure: the `AND` dynamics is more restrictive and a node changes its state only if *both levels* suggest changes; in the `OR` variant *one level* is enough to change the state of an individual.

Finally we propose the following three rules:

(i) `GLOBAL&AND` – global independence and the `AND` rule (see an example in Fig. 3).

With probability p the voter is independent and with $1-p$ behaves like a conformist regardless of the level. In the case of independence the voter changes its state to the opposite one with probability $1/2$ (we automatically change the state of the voter on both levels). In the case of conformity the voter changes its state only when both q-lobbies (i.e., on the first and on the second level) are homogeneous and both have the state opposite to the state of the voter. Therefore only the following changes are possible:

$$\begin{aligned} &\frac{\downarrow\cdots\downarrow}{\downarrow\cdots\downarrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\cdots\downarrow}{\downarrow\cdots\downarrow}\Downarrow,\\ &\frac{\uparrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Downarrow \xrightarrow{1-p} \frac{\uparrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Uparrow,\\ &\frac{\cdots}{\cdots}\Downarrow \xrightarrow{p/2} \frac{\cdots}{\cdots}\Uparrow,\\ &\frac{\cdots}{\cdots}\Uparrow \xrightarrow{p/2} \frac{\cdots}{\cdots}\Downarrow, \end{aligned} \tag{4}$$

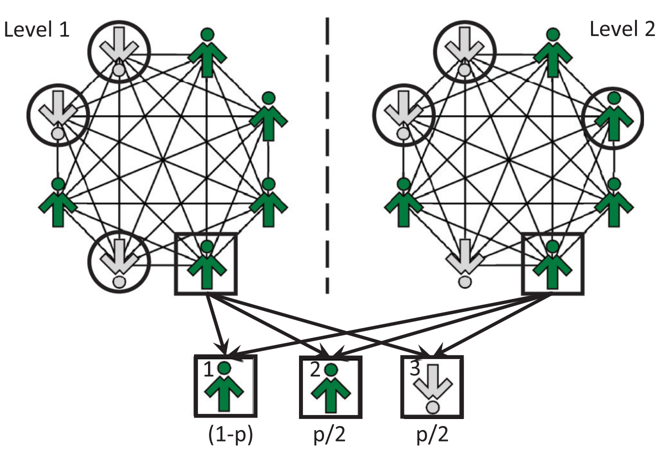

FIG. 3. (Color online) The `GLOBAL&AND` rule: a voter is independent regardless of the level with probability p and is subjected to peer pressure with probability $1-p$ only if q-panels on both levels are self-consistent. In this example, on level 1 the q-lobby (agents in circles) is homogeneous but on level 2 the lobby is not self-consistent. Therefore the voter will not change its state under peer pressure.

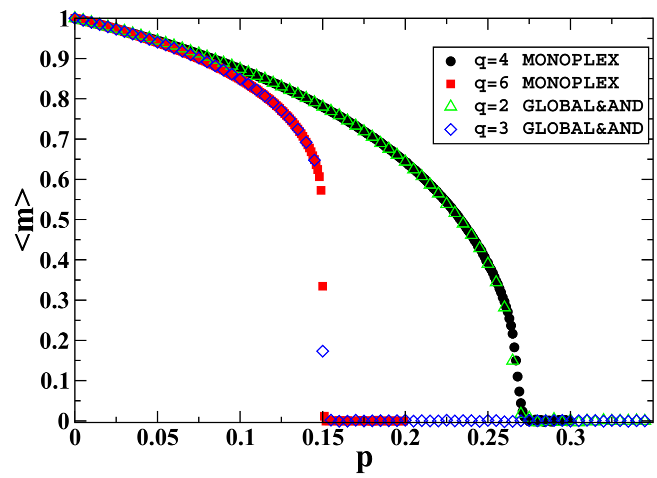

FIG. 4. (Color online) A comparison between the q-voter model on the monoplex clique (full symbols) and the q-voter model with the `GLOBAL&AND` rule on the duplex clique (empty symbols); the `GLOBAL&AND` rule leads to a trivial result, identical with the monoplex case for a doubled value of q. An ensemble average $\langle m\rangle$ of the magnetization, as a function of the stochastic noise p, was obtained by Monte Carlo simulations for a system of size $N=10^4$.

where a single-line arrow represents the state of a node belonging to the q-lobbies; arrows in the numerator represent states of the nodes belonging to the q-lobby chosen on the first level and arrows in the denominator those on second level; the voter's state is marked by a double-line arrow.

We should comment here that the q-voter model with the `GLOBAL&AND` rule and $q=q_2$ on the duplex clique is equivalent to the q-voter on the monoplex clique with $q=q_1=2q_2$ (i.e., a monoplex clique with a q-lobby size twice as large as in the duplex case). This is visible in Fig. 4 where we compare the Monte Carlo simulations obtained for the duplex with those obtained for the monoplex topology.

(ii) `GLOBAL&OR` – global independence and `OR` rule (see an example in Fig. 5).

Here, similarly as in the `GLOBAL&AND` rule, the status of independence is the same for both levels. With probability p the voter is independent and changes its state to the opposite one with probability $1/2$. With probability $1-p$ the voter behaves like a conformist and its state is dependent on both q-lobbies. In contrast to `AND` dynamics, now the voter changes its state to the opposite one even when only one q-lobby is self-consistent and the second is not. In the situation when two q-lobbies are homogeneous but not in agreement, i.e., one q-lobby supports the voter and second suggests to change its state, the voter becomes confused and stays in its old state:

$$\frac{\uparrow\cdots\uparrow}{\downarrow\cdots\downarrow}\Uparrow \xrightarrow[confused]{1-p} \frac{\uparrow\cdots\uparrow}{\downarrow\cdots\downarrow}\Uparrow\,. \tag{5}$$

All situations that lead to change are shown below:

$$\begin{aligned} &\frac{\downarrow\cdots\downarrow}{\downarrow\cdots\downarrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\cdots\downarrow}{\downarrow\cdots\downarrow}\Downarrow,\\ &\frac{\uparrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Downarrow \xrightarrow{1-p} \frac{\uparrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Uparrow, \end{aligned}$$

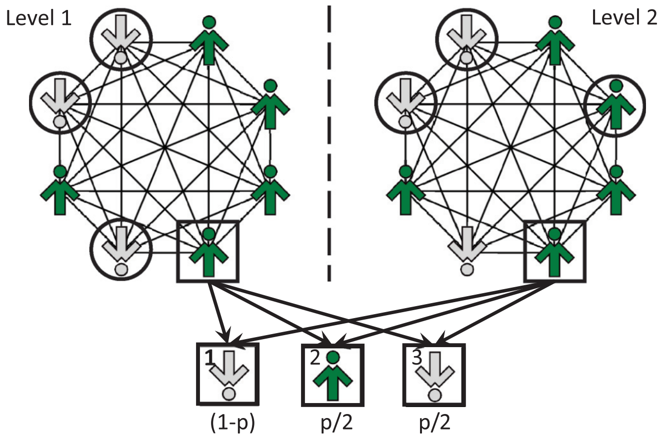

FIG. 5. (Color online) The GLOBAL&OR rule: a voter is independent regardless of the level with probability p and is subjected to peer pressure with probability $1-p$ if at least on one level the q-panel is self-consistent. In this example, on level 1 the q-lobby (agents in circles) is homogeneous and has the state opposite to the state of the voter (an agent in the square). Simultaneously, on level 2 the q-lobby is not self-consistent. Therefore the voter is not confused by two opposite q-lobbies and is influenced by the first q-lobby.

$$\begin{aligned}
&\frac{\downarrow\cdots\uparrow}{\downarrow\cdots\downarrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\cdots\uparrow}{\downarrow\cdots\downarrow}\Downarrow,\\
&\frac{\downarrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Downarrow \xrightarrow{1-p} \frac{\downarrow\cdots\uparrow}{\uparrow\cdots\uparrow}\Uparrow,\\
&\frac{\uparrow\cdots\uparrow}{\downarrow\cdots\uparrow}\Downarrow \xrightarrow{1-p} \frac{\uparrow\cdots\uparrow}{\downarrow\cdots\uparrow}\Uparrow,\\
&\frac{\downarrow\cdots\downarrow}{\downarrow\cdots\uparrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\cdots\downarrow}{\downarrow\cdots\uparrow}\Downarrow,\\
&\frac{\cdots}{\cdots}\Downarrow \xrightarrow{p/2} \frac{\cdots}{\cdots}\Uparrow,\\
&\frac{\cdots}{\cdots}\Uparrow \xrightarrow{p/2} \frac{\cdots}{\cdots}\Downarrow. \qquad (6)
\end{aligned}$$

(iii) LOCAL&AND – local independence and AND rule (see an example in Fig. 6).

In this case the independence is related to the level, i.e., we run dynamics separately on each level. It means that a voter is independent on the first level with probability p and with probability $1-p$ behaves as a conformist; it is under the influence of the q-lobby on this level. The same situation is on the second level, where regardless of the first level we choose if the voter behaves independently or conforms the q-lobby on the second level. Finally we change the state of the voter only when both separated dynamics result in the same state.

We do not consider the LOCAL&OR rule because of the difficulty of such a concept related to social unreality and algorithmic ambiguity.

We would like to stress that we are aware that the topology we consider here is very artificial and not suitable to describe many real social systems. We have chosen this topology mainly because it allows for analytical treatment (see Sec. IV). Moreover, until now the q-voter model with independence has been systematically analyzed only on the complete graph [13,20], and a duplex clique seemed to be a natural extension that would allow for a straightforward

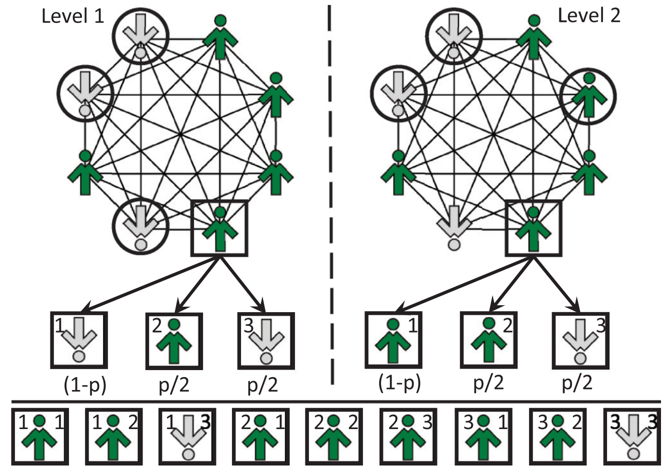

FIG. 6. (Color online) The LOCAL&AND rule: a voter is independent separately on each level with probability p and subjected to peer pressure separately on each level with probability $1-p$. It means that on each level independently the voter (in the square) can be in one of three possible states: $+1$ with probability $p/2$, -1 with probability $p/2$, and in a state suggested by the q-lobby (in circles) with probability $1-p$. In this example, the q-lobby is homogeneous on the first level and therefore it influences the voter. On level 2 the q-lobby is not self-consistent and therefore there is no peer pressure. Finally, there are nine possible pairs that represent states on the first and on the second level; see the numbers in the corners of squares in the bottom line. The voter changes its state only if states, obtained independently on each level, are the same; see all possible final states of the voter in the bottom line.

comparison with the previously obtained results. On the other hand, the ideas discussed here allow us to investigate the model on an arbitrary multiplex, because each level can potentially be represented by a different complex network. Results for selected network topologies will be also discussed in Sec. VIII, although they are not the primary goal of this paper.

IV. THE TIME EVOLUTION

The aim of this section is to derive equations that describe the time evolution of the system for each of three considered rules. Let us denote by $N_{\uparrow}(t)$ the number of voters in the $+1$ state (up-spins) at time t and by $N_{\downarrow}(t)$ the number of voters in the -1 state (down-spins). The total number N of spins in a system does not change and we can define the concentration of up-spins at time t as

$$c(t)=\frac{N_{\uparrow}(t)}{N}. \qquad (7)$$

Since all individuals keep the same state on both levels and we consider a duplex clique, we can simplify our analysis by considering the concentration $c(t)$ only on one level. However, we need to stress that the changes of the state of the node occur under the influence of both levels. In a single time step Δ_t three scenarios are possible: the number of up-spins $N_{\uparrow}(t)$ will either increase by 1, decrease by 1, or remain constant. Simultaneously the concentration $c(t)$ increases or decreases by $\frac{1}{N}$ or remains constant:

$$\begin{aligned}\gamma^+(c) &= \Pr\left\{c(t+\Delta_t) = c(t) + \frac{1}{N}\right\},\\ \gamma^-(c) &= \Pr\left\{c(t+\Delta_t) = c(t) - \frac{1}{N}\right\},\\ \gamma^0(c) &= \Pr\{c(t+\Delta_t) = c(t)\} = 1 - \gamma^+(c) - \gamma^-(c).\end{aligned} \tag{8}$$

The time evolution of the average concentration is given by the rate equation:

$$\langle c(t+\Delta_t)\rangle = \langle c(t)\rangle + \frac{1}{N}[\gamma^+(c) - \gamma^-(c)], \tag{9}$$

where the exact formulas for probabilities $\gamma^+(c)$ and $\gamma^-(c)$ depend on the applied rule. In the following part of the paper, we use the abbreviated notation replacing $\gamma^+(c)$ by γ^+, $\gamma^-(c)$ by γ^-, and $c(t)$ by c. Explicit forms of probabilities γ^+,γ^- are the following.

(i) `GLOBAL&AND`:

$$\begin{aligned}\gamma^+ &= (1-p)(1-c)c^{2q} + p(1-c)/2,\\ \gamma^- &= (1-p)c(1-c)^{2q} + pc/2.\end{aligned} \tag{10}$$

The first component describes conformity, where the change of the state is possible only when two lobbies of size q each (i.e., the total number of agents is equal to $2q$) possess the opposite state than the state of the voter. The second component is responsible for the change due to independence.

(ii) `GLOBAL&OR`:

$$\begin{aligned}\gamma^+ &= (1-p)(1-c)\left[2\sum_{k=1}^{k=q-1}\binom{q}{k}c^{q+k}(1-c)^{q-k} + c^{2q}\right]\\ &\quad + \frac{p(1-c)}{2},\\ \gamma^- &= (1-p)c\left[2\sum_{k=1}^{k=q-1}\binom{q}{k}(1-c)^{q+k}c^{q-k} + (1-c)^{2q}\right]\\ &\quad + \frac{pc}{2}.\end{aligned} \tag{11}$$

Since in the `OR` case agreement just one lobby is needed, the sum in the above equations reflects all possible states in which all agents in the q-lobby on the one level possess the state opposite to the state of the voter, and simultaneously the q-lobby on the second level is not homogeneous. The `GLOBAL&OR` rule for $q = 2$ indicates change of the voter's state if three or four of the four agents (two from each level) possess the same state, and therefore it is equivalent to majority rule [40–43]. For $q > 2$ majority rule is not enough since changes are possible only when at least one lobby is homogeneous, e.g.,

$$\frac{\downarrow\downarrow\downarrow}{\downarrow\uparrow\uparrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\downarrow\downarrow}{\downarrow\uparrow\uparrow}\Downarrow. \tag{12}$$

This fact is direct reason why in the following example there is no change of state:

$$\frac{\downarrow\uparrow\downarrow}{\downarrow\uparrow\downarrow}\Uparrow \xrightarrow{1-p} \frac{\downarrow\uparrow\downarrow}{\downarrow\uparrow\downarrow}\Uparrow. \tag{13}$$

(iii) `LOCAL&AND`:

$$\begin{aligned}\gamma^+ &= (1-p)^2(1-c)c^{2q} + p(1-p)(1-c)c^q + \frac{p^2(1-c)}{4},\\ \gamma^- &= (1-p)^2c(1-c)^{2q} + p(1-p)c(1-c)^q + \frac{p^2c}{4}.\end{aligned} \tag{14}$$

Here we have three components: the first describes the situation when the voter behaves like a conformist on both levels, the last one corresponds to the case where on both levels the voter is independent. The second term in Eq.(14) is a mixed one: the voter behaves as a conformist $[(1-p)c^q]$ on one level and simultaneously it is independent on the second level ($p/2$). We multiple this middle term by 2 since this situation can appear in two configurations: conformist on the first level and independent on the second and vice versa.

V. RESULTS

Solving analytically the rate equation (9) in general (i.e., for arbitrary q) is a difficult task. However, it is easy to obtain a numerical solution by iterating Eq. (9). In such a way we can obtain the time evolution of the average concentration $\langle c(t)\rangle$, as well as the stationary value $\langle c\rangle$. The magnetization $m(t)$ defined by Eq. (2) is directly related to the concentration $c(t)$,

$$m(t) = \frac{N_\uparrow(t) - N_\downarrow(t)}{N} = 2c(t) - 1, \tag{15}$$

and therefore using the rate equation (9) we can easily find also the average magnetization. Independently, we can obtain results by conducting Monte Carlo simulations and calculating the ensemble average of the magnetization defined by Eq. (3).

Relations between the average magnetization in the stationary state $\langle m\rangle$ and the stochastic noise p, obtained by two methods (numerical solutions of analytical formulas and Monte Carlo simulations), are presented in Figs. 7, 8, and 9. It is seen that the agreement between the Monte Carlo (MC) results obtained for system size $N = 10^4$ and the numerical

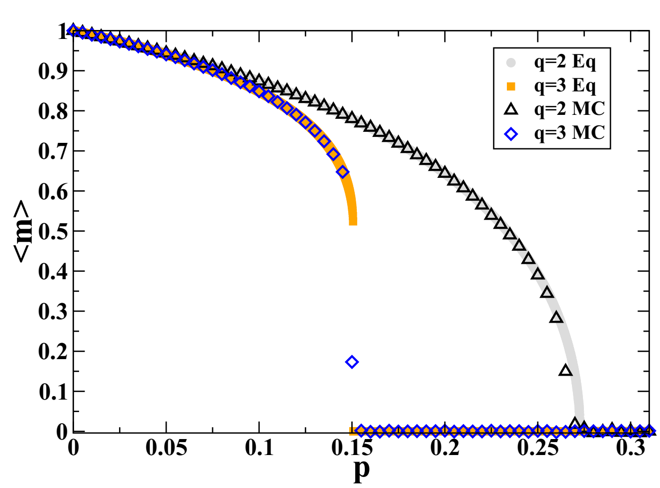

FIG. 7. (Color online) The average magnetization $\langle m\rangle$ as a function of the stochastic noise p for the `GLOBAL&AND` rule on the duplex clique. Monte Carlo results (empty symbols) were obtained for a system of size $N = 10^4$ and averaged over 10^3 samples. The numerical solutions of Eq. (9) are marked with full symbols.

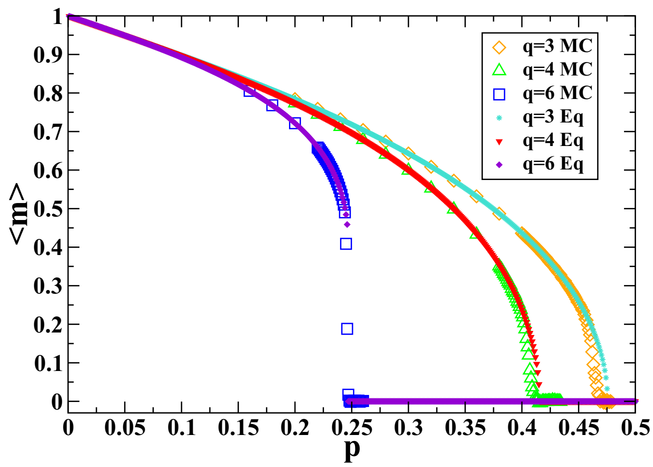

FIG. 8. (Color online) The average magnetization $\langle m\rangle$ as a function of the stochastic noise p for the `GLOBAL&OR` rule on the duplex clique. Monte Carlo results (empty symbols) were obtained for a system of size $N = 10^4$ and averaged over 10^3 samples. The numerical solutions of Eq. (9) are marked with full symbols.

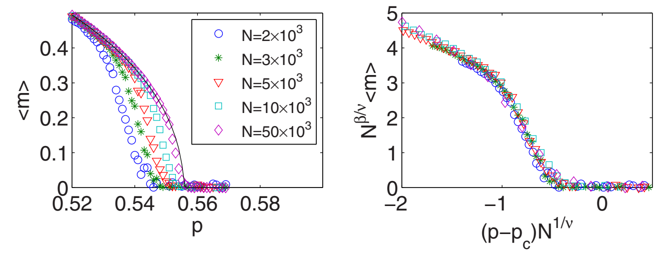

FIG. 10. (Color online) The average magnetization $\langle m\rangle$ as a function of stochastic noise p for the `LOCAL&AND` rule on the duplex clique for $q = 3$ and several system sizes N. The solid line represents numerical results obtained from Eq. (9) for the infinite system. Rescaled results using the finite-size scaling technique are presented in the right panel. In this case the critical value of independence can be estimated as $p_c \approx 0.55$ and coincides with the numerical result for the infinite system. Critical exponents are $\beta \approx 0.5$ and $\nu \approx 2.0$ for this version of the model with $q = 3$.

solution of Eq. (9) for the infinite system size ($N \to \infty$) is satisfactory.

However, a small gap between MC and numerical results can be seen. One could ask if the gap is an artifact of the finite system size or not. As can be seen in Fig. 10 it is indeed a finite-size effect. In this figure we present results for different system sizes in the case of the `LOCAL&AND` rule with $q = 3$. Obviously, increasing the system size we approach the numerical results for the infinite system. Moreover, using the finite-size scaling technique we are able to identify the "real" critical value of independence that coincides with the value obtained from Eq. (9) for the infinite system and to determine critical exponents ν and β. However, the exact values of these exponents depend not only on the version and parameters of the model but also on the network structure, and this dependence does not seem to be trivial [44]. Therefore, another task that could be considered in the future is the exact relation between the finite-size scaling exponents and the parameters of the model.

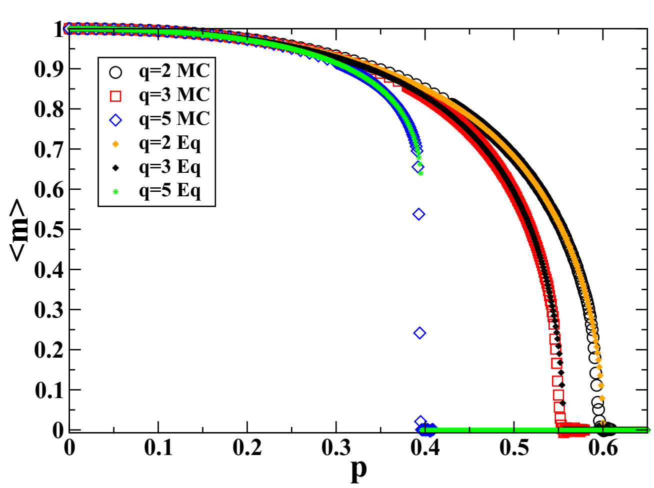

FIG. 9. (Color online) The average magnetization $\langle m\rangle$ as a function of the stochastic noise p for the `LOCAL&AND` rule on the duplex clique. Monte Carlo results (empty symbols) were obtained for a system of size $N = 10^4$ and averaged over 10^3 samples. The numerical solutions of Eq. (9) are marked with full symbols.

VI. PHASE TRANSITIONS

Due to the Landau theory, to describe any kind of a phase transition we can introduce the quantity that measure the degree of order (order parameter) [45,46]. Although originally Landau theory was created to describe continuous phase transitions [45], the theory can be used also in the case of discontinuous phase transitions [46]. An order parameter, introduced to distinguish between two phases, is equal to 1 in the completely ordered state, decreases as a function of the deviation from the order, and becomes zero in the disordered phase. Therefore for our system the natural choice of the order parameter is an average magnetization $\langle m\rangle$.

It is seen in Figs. 7, 8, and 9 that for all three rules `GLOBAL&AND`, `GLOBAL&OR`, and `LOCAL&AND`, the system undergoes the phase transition. Below the transition point $p = p_c$, the average magnetization $\langle m\rangle$ (order parameter) is not equal to zero, and above the transition point $\langle m\rangle = 0$. For `GLOBAL&AND` (see full symbols in Fig. 7) the transition changes its type from continuous to discontinuous at $q = 3$, which corresponds to $q = 6$ for the monoplex clique and thus agrees with results obtained in [13]. Analogously, for `GLOBAL&OR` (see full symbols in Fig. 8) the transition changes its type also at $q = 6$. However, for `LOCAL&AND` (see full symbols in Fig. 9), the transition changes its type from continuous to discontinuous already at $q = 5$. Moreover, the transition point is much higher than for the remaining two rules. The aim of this section is to determine the relation between the threshold value p_c and parameter q for all three rules and to better understand the nature of the observed phase transitions. We would like to stress here that, although at this point the claims of continuous or discontinuous transitions are solely supported by Figs. 7, 8, and 9, a more thorough analysis is conducted below to support these claims.

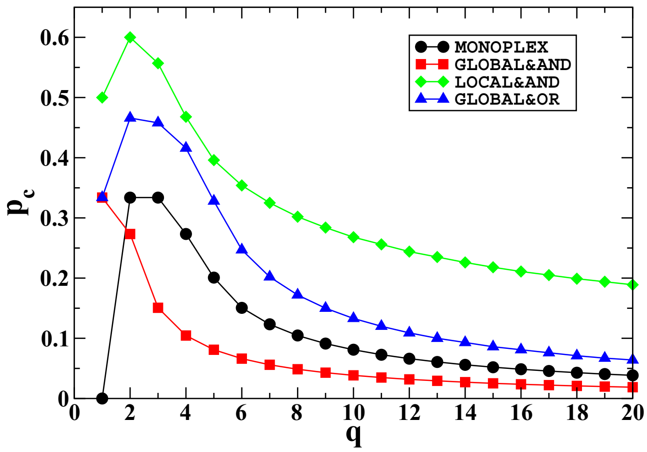

FIG. 11. (Color online) The relation between the critical value of the stochastic noise p_c and parameter q for different variants of the model.

We can obtain the the transition point $p = p_c$ directly from the Landau definition of an order parameter $\langle m \rangle$ using numerical solutions of Eq. (9) or Monte Carlo simulations. Because, as seen in Figs. 7, 8, and 9, the agreement between Monte Carlo and numerical stationary solutions of Eq. (9) is very good, we determine p_c from Eq. (9), which is not only much faster but also a more accurate method. Figure 11 shows the relation between the critical value of noise p_c and the size of the q-lobby.

It is obvious why `GLOBAL&AND` gives always a lower value of $p_c(q)$ than `GLOBAL&OR`. In the former case, unanimous q-panels have to be chosen on both levels, which is less probable than choosing a unanimous q-panel only on one level. Therefore the order is more easily destroyed for the `GLOBAL&AND` rule, which results in a lower value of $p_c(q)$. Only for $q = 1$ is the critical value of noise, p, the same for both rules; in this case, for both rules the change of the voter's state can take place only if the neighbors, chosen from the first and the second level, possess the same state and the opposite of the voter's state. Otherwise, for the `GLOBAL&AND` rule a voter is confused and does not change. It is also seen that the value of $p_c(q)$ for `GLOBAL&AND` is exactly the same as $p_c(2q)$ for a monoplex, for which the critical value of the noise for $q \leqslant 5$ has been derived analytically [13]:

$$p_c(q) = \frac{q-1}{q-1+2^{q-1}}. \tag{16}$$

The highest value of $p_c(q)$ is obtained for `LOCAL&AND`. This case is not so easy to analyze heuristically and is certainly the most interesting, among the considered rules. For both `GLOBAL` rules the multiplex could be in fact replaced by the monoplex network. In the case of `GLOBAL&AND`, as already mentioned, we could simply consider the q-voter on the monoplex with doubled size of the q-lobby. The `GLOBAL&OR` is less trivial but still could be probably reformulated in terms of the q-voter model with the threshold on the monoplex [20]. The case of `LOCAL` independence is not only less trivial, but also the most interesting from the social point of view. It should be remembered that conformity (and simultaneously independence) is relative, i.e., individuals always conform in respect to the particular social group and there are many factors that influence the level of conformity [5,47–49]. It means that the same individual may conform to one group and behave independently with respect to another. For example it has been shown, on the basis of various social experiments, that the level of conformity is much higher in the face-to-face condition than in computer-mediated communication [47,48]. Hence the idea of local independence is highly justified in modeling social systems. Moreover, we do not see any possibility to replace a duplex by a monoplex network in this case, whereas in the other two cases such a mapping is straightforward, as already discussed.

Therefore, we will now concentrate on the `LOCAL&AND` rule and discuss the phase transition more thoroughly in this case. First of all let us notice that

$$F = \gamma^+ - \gamma^- \tag{17}$$

can be treated, analogously as in [13], as an effective force: γ^+ drives the system to the state "spins up," while γ^- to "spins down." Therefore, inserting explicit forms of γ^+,γ^- from Eq. (14), we calculate also an effective potential:

$$\begin{aligned} V &= -\int F\,dc \\ &= p(1-p)\{q[(1-c)c^{q-1} + c(1-c)^{q-1}] \\ &\quad - [c^q - (1-c)^q]\} + (1-p)^2\{2q[(1-c)c^{2q-1} \\ &\quad + c(1-c)^{2q-1}] - [c^{2q} - (1-c)^{2q}]\} - p^2/2. \end{aligned} \tag{18}$$

To find the critical value of the stochastic noise p_c and the threshold value $\tilde{q}$, above which the transition becomes discontinuous, we could now use the Landau approach, analogously

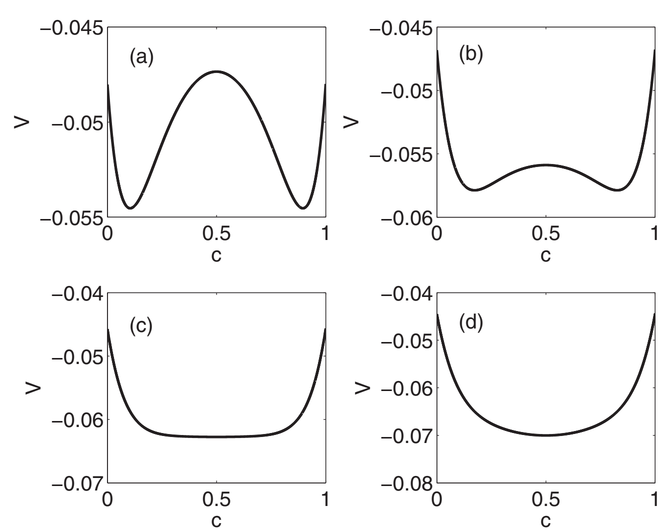

FIG. 12. An effective potential V, given by Eq. (18), as a function of concentration c for the `LOCAL&AND` rule and $q = 4$. (a) For small values of noise p (here $p = 0.4$) the potential has two minima that correspond to ordered states (i.e., $c \neq 1/2$ and simultaneously $m \neq 0$). (b) With increasing p (here $p = 0.44$), minima are getting shallower and areapproaching each other. (c) Eventually ($p \approx 0.47$) they form a single minimum that corresponds to the new disordered phase. (d) Further increase of p (here $p = 0.5$) results in deepening the minimum. This is a typical behavior for a *continuous phase transition*.

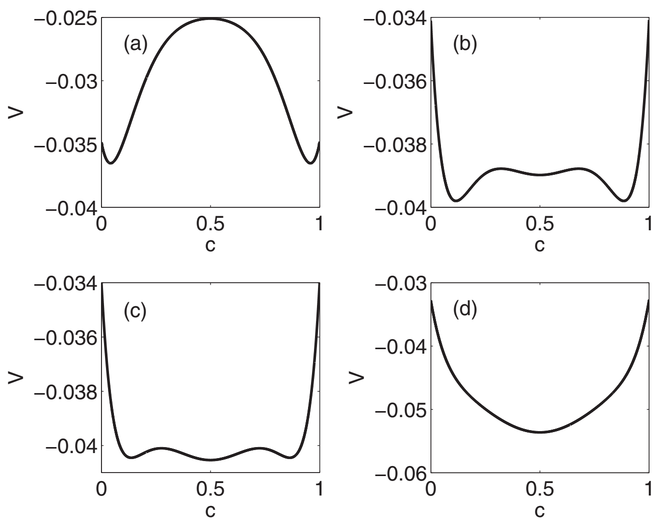

FIG. 13. An effective potential V, given by Eq. (18), as a function of c for the LOCAL&AND rule and $q = 5$. (a) For small values of noise p (here $p = 0.3$) the potential has two minima that correspond to ordered states (i.e., $c \neq 1/2$ and simultaneously $m \neq 0$). (a) For larger p (here $p = 0.38$) the third minimum (that corresponds to the new disordered phase) appears. Initially [for $p \in (p_1^*, p_2^*)$] the minima, that correspond to an ordered phase, are deeper than the middle one; i.e., the disordered state is metastable. (c) For $p = p_2^* \approx 0.388$ all three minima are equally deep; i.e., ordered and disordered states are equally probable, which corresponds to the *discontinuous phase transition*. For $p \in (p_2^*, p_3^*)$ the potential has still three minima, but now two ordered states are metastable. (d) Finally, for $p \in (p_3^*, 1)$ (here $p = 0.45$) the potential has only one minimum that corresponds to the disordered phase. Between spinodal lines, i.e., for $p \in (p_1^*, p_3^*)$ one can expect hysteresis, and indeed it was found in Monte Carlo simulations (see Fig. 14).

as in [13]. To do this we first rewrite the potential (18) in terms of magnetization m, using relation (15), and then expand it into a power series around $m = 0$. Unfortunately, the form of the potential is much more complex in this case than for the q-voter model on the monoplex [13]. Therefore all formulas are much longer and more difficult to analyze. To understand the nature of the phase transition, it is much easier and more illustrative to draw potential V as a function of c for different values of p and q (see Figs. 12 and 13).

For $q < \tilde{q} = 5$ (see Fig. 12) the potential, given by Eq. (18), behaves typically for a continuous phase transition. Below the transition point the potential has two minima that correspond to ordered states (i.e., $c \neq 1/2$ and simultaneously $m \neq 0$). With increasing p, minima are getting shallower and approaching each other. Eventually they form a single minimum that corresponds to the new disordered phase. For $q \geqslant \tilde{q}$ the potential indicates a discontinuous phase transition. For small values of noise p the potential has two minima that correspond to ordered states (i.e., $c \neq 1/2$ and simultaneously $m \neq 0$). For larger p the third minimum (that corresponds to the new disordered phase) appears. Initially [for $p \in (p_1^*, p_2^*)$] the minima that correspond to an ordered phase are deeper than the middle one, i.e., the disordered state is metastable. For $p = p_2^*$ all three minima are equally deep; i.e., ordered and disordered states are equally probable, which corresponds

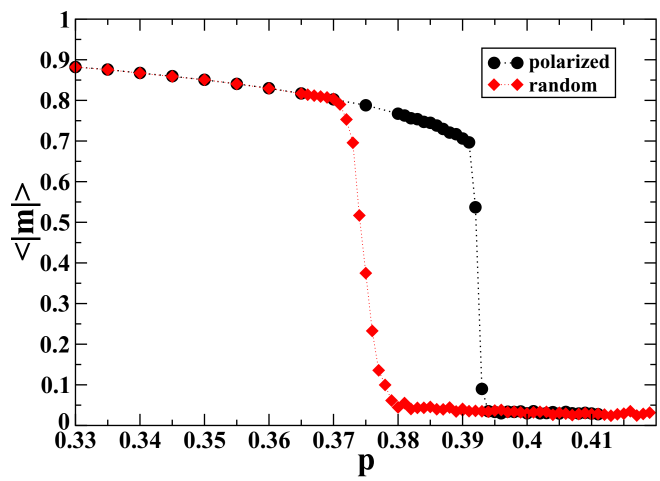

FIG. 14. (Color online) The average magnetization $\langle |m| \rangle$ as a function of the stochastic noise p for the LOCAL&AND rule and $q = 5$ obtained from Monte Carlo simulations. Two different initial states were considered: "polarized," i.e., ordered state with $m = 1$, and "random," i.e., disordered state with $m = 0$. As expected for a discontinuous phase transition (see Fig. 13), hysteresis is observed.

to the discontinuous phase transition. For $p \in (p_2^*, p_3^*)$ the potential has still three minima, but now two ordered states are metastable. Finally, for $p \in (p_3^*, 1)$ the potential has only one minimum that corresponds to the disordered phase. Between the spinodal lines, i.e., for $p \in (p_1^*, p_3^*)$, one can expect hysteresis, and indeed it was found in Monte Carlo simulations (see Fig. 14).

The evidence of the first-order transitions for $q = 5$ can be obtained also from MC simulations using the finite-size scaling technique. We should be able to observe a jump of the order parameter at the transition point, and this jump should tend to a positive constant which is equal to the jump $m(p^*)$ for the infinite system. Until now—see Figs. 7, 8, and 9—we have averaged results over samples. Although, as we have checked, an averaging over samples gives the same result as an averaging

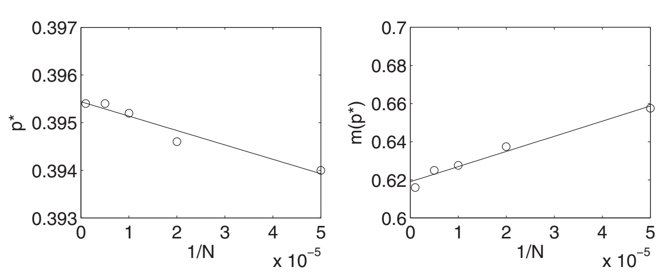

FIG. 15. Results for the LOCAL&AND rule and $q = 5$ from the Monte Carlo simulations. Results were averaged over 10^4 samples after the thermalization time of 10^4 Monte Carlo steps (MCS). Left panel: dependence of the transition point $p^*(N)$ on the system size N. It is seen that for $N \to \infty$ the transition point $p^*(N)$ approaches $p^* \approx 0.3955$, which agrees with the value obtained from the rate equation for the infinite system. Right panel: a jump of the order parameter at the transition point $m(p^*)$ as a function of the system size N. It is seen that for $N \to \infty$ the jump approaches a positive constant, which confirms a discontinuous phase transition.

over time, the jump of the order parameter is seen more clearly in the case of the time average, especially for small systems. Therefore, to determine the size of the jump $m(p^*)$ for various sizes N, we have used the time average. It is seen in Fig. 15 that indeed for $N \to \infty$ the jump approaches a positive constant, which confirms a discontinuous phase transition.

VII. *L*-LEVEL CLIQUE

The result that a switch from a continuous to a discontinuous phase transition occurs for $q = 5$ instead of $q = 6$ is obtained mainly from the Landau approach. It is confirmed by MC simulations, but an intuitive understanding of this fact is still missing. One possible explanation is that adding another level is similar to adding another dimension. In equilibrium statistical mechanics it is common that systems exhibiting a discontinuous phase transition in high space dimensions may display a continuous transition below a certain critical dimension [50]. An analogous phenomenon could be observed here. This would explain the fact that the system with $q = 5$ in a "higher dimension" undergoes a discontinuous phase transition instead of a continuous one as in a "lower dimension." To validate this intuition we now consider the `LOCAL&AND` rule on an L-level clique. In such a case the network consists of L distinct levels, each of which is represented by a complete graph (i.e., a clique) of size N, and the probabilities of gain and loss can be written as

$$\gamma^+ = (1-c)\left[\sum_{i=0}^{i=L}\binom{L}{i}p^{L-i}\left(\frac{1}{2}\right)^{L-i}[(1-p)c^q]^i\right],$$
$$\gamma^- = c\left[\sum_{i=0}^{i=L}\binom{L}{i}p^{L-i}\left(\frac{1}{2}\right)^{L-i}[(1-p)(1-c)^q]^i\right]. \tag{19}$$

For $L = 1$ the above equation reduces to the rate equation formulated by Nyczka *et al.* [20] and for $L = 2$ the equation is equivalent to Eq. (14). The steady value of concentration [or equivalently magnetization, if we use relation (15)] can be obtained from iterations of the rate equation (9), as we previously have done for $L = 2$, or alternatively by numerically solving the equation

$$\gamma^+ - \gamma^- = 0. \tag{20}$$

It turns out that, for $q < \widetilde{q}(L)$ and $p < p_c$, Eq. (20) has two stable solutions $m_+ = -m_- \neq 0$, whereas for $p > p_c$ it as one stable solution $m_0 = 0$. For $q \geqslant \widetilde{q}$ the situation is more complicated. For small values of noise ($p < p_1^*$) there are again two stable solutions $m_+ = -m_- \neq 0$, whereas for large values of noise ($p > p_3^*$) one stable solution $m_0 = 0$. However, for $p \in (p_1^*, p_3^*)$ there are three stable solutions $m_+ = -m_- \neq 0, m_0 = 0$ and two unstable ones m_1 and m_2, where $m_+ > m_1 > m_0$ and $m_0 > m_2 > m_-$. Therefore the final state of the system depends on the initial state (hysteresis):

(1) if the initial value of magnetization $m(0) > m_1$ then the system reaches the final ordered state with magnetization $m(\infty) = m_+$,

(2) if $m(0) < m_2$ then $m(\infty) = m_-$,

(3) if $m_2 < m(0) < m_1$ then the system reaches the final state with $m(\infty) = 0$.

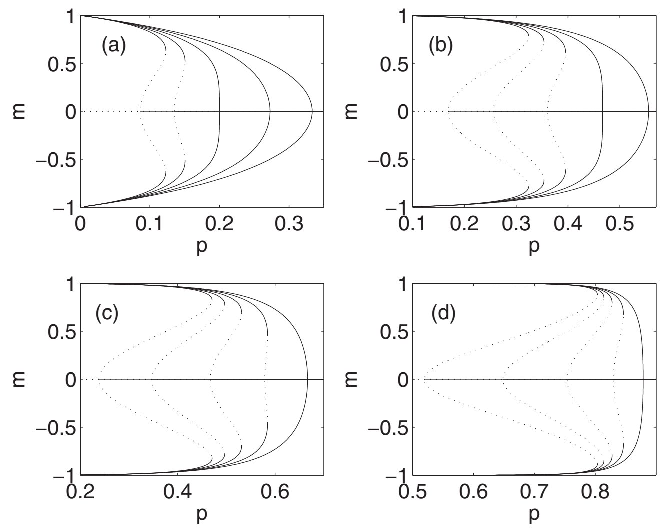

FIG. 16. Relation between the steady value of magnetization m obtained from Eq. (20) with probabilities γ^+, γ^- given by Eq. (19) for several values of L: (a) $L = 1$, (b) $L = 2$, (c) $L = 3$, and (d) $L = 10$. On each panel the relation for $q = 7,6,5,4,3$ is presented (from left to right). It is seen that for some values of q, specifically for $q \geqslant \widetilde{q}(L)$, where (a) $\widetilde{q}(L = 1) = 6$, (b) $\widetilde{q}(L = 2) = 5$, (c) $\widetilde{q}(L = 3) = 4$, and (d) $\widetilde{q}(L > 3) = 4$, there is an interval $p \in (p_1^*, p_3^*)$ in which Eq. (20) has five solutions: three stable (solid lines) $m_+ = -m_- \neq 0, m_0 = 0$ and two unstable (dashed lines) m_1 and m_2, where $m_+ > m_1 > m_0$ and $m_0 > m_2 > m_-$.

This means that for $q \geqslant \widetilde{q}$ the system undergoes a discontinuous phase transition. It is seen in Fig. 16 that $\widetilde{q}(L = 1) = 6$, $\widetilde{q}(L = 2) = 5$, and $\widetilde{q}(L \geqslant 3) = 4$. Hence, as expected, the critical value $\widetilde{q}$—at which the transition switches from a continuous to a discontinuous one—decreases with the number of levels L, but only to the threshold value of $L = 3$.

Now we would like to comment on the social implications of the obtained results, namely that the threshold size of the group below which the transition is always continuous is equal to $\widetilde{q}(L \geqslant 3) = 4$. The "mysterious" group of four repeatedly shows up in social sciences. For example, the optimal group size for discussions, collaboration, etc., has been an issue of interest for years, mainly from the management science point of view. Already in the 1970s it was concluded from a cross-sectional study that the optimal team size was between four and five members [51]. Moreover, the size of the group is also very important in the context of social influence. It has been observed in a number of social experiments that for social influence it is not only essential for the majority to be unanimous but also of a sufficient size of about 3–5 people. Increasing the size of the majority will have no additional impact [52]. Certainly, this convergence may be accidental. The question is if the threshold value, $\widetilde{q}$, decreases in reality with the number of levels of the social network. To the best of our knowledge, this question has not been investigated empirically to date. However, there is a more basic question we have not answered yet: Will the regime-switching behavior be observed for other topologies and not only for a complete graph? Therefore in Sec. VIII we will consider other duplex topologies.

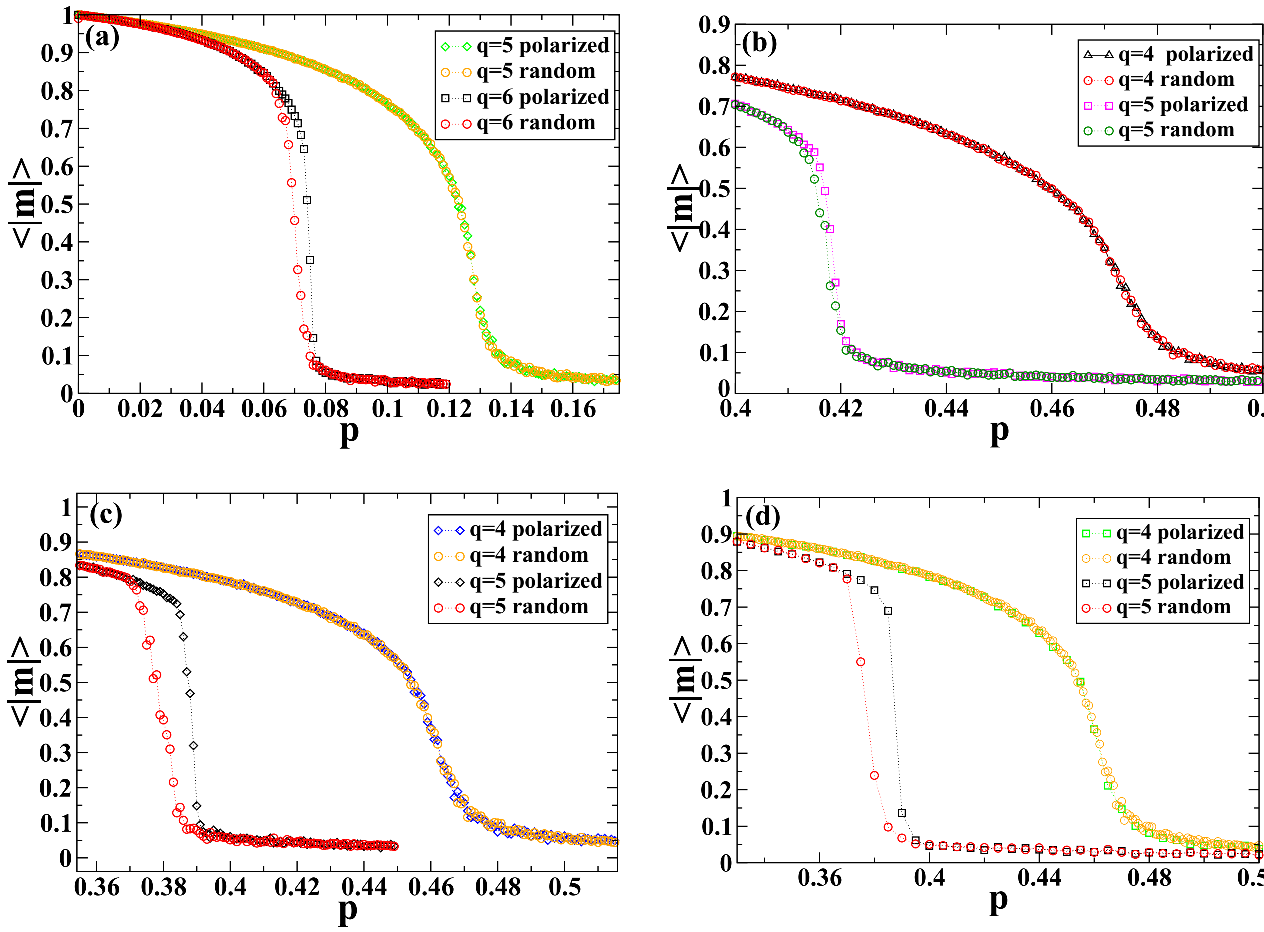

FIG. 17. (Color online) Average magnetization $\langle|m|\rangle$ as a function of the stochastic noise p for the LOCAL&AND rule. Two different initial states are considered: "polarized," i.e., an ordered state with $|m(0)| = 1$, and "random," i.e., a disordered state with $m(0) = 0$. (a) Monoplex network BA network, (b) duplex ER network with the absence of interlayer correlations, (c) duplex BA networks with positive $\langle r_{lay}\rangle = 0.82$, and (d) negative interlayer correlations $\langle r_{lay}\rangle = -0.13$ are presented. Monte Carlo results were obtained for a system of size $N = 5000$ and averaged over 200 samples. The average interlayer correlation was averaged over 200 samples.

VIII. DUPLEX NETWORKS

In this section we go beyond the duplex clique to check to what extent our results are universal. To test if $\widetilde{q} = 6$ is the universal value also for other monoplex topologies we have conducted Monte Carlo simulations for the q-voter model on monoplex Barabási-Albert (BA) scale-free networks and monoplex Erdős-Rényi (ER) graphs. Indeed, the switch from a continuous to a discontinuous phase transition is observed at $q = \widetilde{q} = 6$ for both topologies (the hysteresis is visible in Fig. 17), which indicates that $\widetilde{q}$ does not depend on the degree distribution.

Now the question is if adding another layer will shift $\widetilde{q} = 6$ to $\widetilde{q} = 5$, as in the case of a complete graph. Therefore we consider a duplex which consists of two distinct levels, each represented by a different BA network (or a different ER graph). We consider not only different degree distributions on each level (power-law or Poisson) but also different interlayer correlations (i.e., Pearson correlation coefficients between the degrees of nodes on the first and the second level) [53,54]. Consequently, we analyze selected duplex networks that represent three different classes: (i) a duplex ER network with the absence of interlayer correlations, (ii) a duplex BA network with positive correlations, and (iii) a duplex BA network with negative correlations. In all considered cases, we observe the switch from a continuous to a discontinuous phase transition at $\widetilde{q} = 5$, analogously as for the duplex clique. It is shown in Fig. 17 that the hysteresis is observed in all cases for $q = 5$, whereas it is not present for $q = 4$. These results suggest that the relation between the type of the phase transition and the critical value, $\widetilde{q}$, does not depend on a particular topology but only on the number of levels.

IX. CONCLUSIONS

We have generalized the q-voter model with independence (stochastic noise) p to a duplex clique, i.e., a network that consists of two distinct levels (layers), each of which is represented by a complete graph (i.e., a clique) of size N. Levels represent two different communities (e.g., Facebook and school class), but are composed of exactly the same people: each node possesses a counterpart node in the second level. Such an assumption reflects the fact that we consider fully overlapping levels, which is an idealistic scenario. We also assume that each node possesses the same state on each level, which means that the society consists of nonhypocritical individuals only. We have considered two criteria of level dependence: one related to the status of independence (`GLOBAL` vs `LOCAL`) and one related to peer pressure (`AND` vs `OR`). The `GLOBAL` rule means that an agent is independent on both levels, but the `LOCAL` rule admits a situation where a person is independent in one clique but not in the other. Furthermore, the `AND` dynamics is more restrictive and a node changes its state only if *both levels* suggest changes; in the `OR` variant *one level* is enough to change the state of an individual.

For all three considered rules (`GLOBAL&AND`, `GLOBAL&OR`, and `LOCAL&AND`), the system undergoes a continuous order-disorder phase transition at $p = p_c(q)$ for $q < \widetilde{q}$ and a discontinuous for $q \geqslant \widetilde{q}$, where p_c and $\widetilde{q}$ are rule dependent. The `GLOBAL&AND` rule leads to a trivial result, identical with the monoplex case for a doubled value of q. For the `GLOBAL&OR` dynamics, p_c is larger than for the monoplex network. However, $\widetilde{q}$ is identical with the monoplex case, i.e., $\widetilde{q} = 6$. In contrast to the other two rules, we find a qualitative change for the `LOCAL&AND` rule, as the phase transition becomes discontinuous for $\widetilde{q} = 5$. The case of `LOCAL` independence is not only less trivial, but also more interesting and better justified from the social point of view. In particular, it has been shown that the level of conformity during face-to-face communication is significantly higher than during computer-mediated communication such as the Internet [47,48].

This suggests that the `LOCAL&AND` rule is the most suitable for real social systems. Certainly it could be further developed by introducing different values of noise on each level. The simplistic duplex clique topology, as introduced in this paper, can be also modified to obtain a more general network. For instance, one could consider partially overlapping cliques, where some nodes possess no counternode on the second level. Unfortunately, these modifications significantly complicate the model by introducing additional parameters and therefore are beyond the scope of this paper. However, even considering such a simple model as the one presented here, we can observe that a multiplex network can introduce significant differences in opinion dynamics.

From the physical point of view, the `LOCAL&AND` rule is the most interesting. For this rule a qualitative change is observed: the threshold value of q, below which the phase transition is continuous shifted from $\widetilde{q} = 6$ for monoplex networks (including complete and random graphs, as well as Barabasi-Albert networks) to $\widetilde{q} = 5$ for duplex structures. The classification of phase transitions is one of the most exiting topics in the field of statistical physics. One of the best examples of interest is a recent hot debate on the type of the phase transition in so-called explosive percolation [55–57]. However, for the q-voter model it seems to be clear that there is a switch from a continuous to a discontinuous phase transition, because it is analytically solvable at least in the case of a complete graph [13]. The question that arose here is why for a duplex network the switch from a discontinuous to a continuous phase transition appears for lower q, which means that an additional level facilitates a discontinuous phase transition.

The mechanism that leads to a discontinuous transition, that manifests as a jump of the order parameter, is usually related to fluctuations. It is known that by increasing the number of interacting neighbors the fluctuations are diminished and the transitions become sharper [13,58,59]. In our opinion a similar mechanism is observed here, although the ultimate, intuitive understanding of the phenomena is still missing. There are other possible causes of a discontinuous phase transition. Sometimes a switch from a continuous to a discontinuous phase transition is observed for a larger number of states (like in the Potts model [60] or in a model of tactical voting [61]). A switch from a continuous to a discontinuous phase transition has been also observed in the SIR (susceptible-infected-removed) model with cooperative co-infection [62], in which the number of states for each individual is 9. It has been suggested in this paper that the discontinuity of phase transitions results from the fact that the "basic reproduction ratio" (which applies to infinitesimally small initial epidemic seeds) is smaller than the reproduction ratio that applies when the fraction of infected is finite.

It seems that in our case an additional level plays a similar role to an additional dimension, and therefore an increase of the number of levels supports discontinuity. In equilibrium statistical mechanics, it is common that systems which exhibit a discontinuous phase transition in high space dimensions may display a continuous transition below a certain upper critical dimension [50]. An analogous phenomenon could be observed here. This intuition has been confirmed by the results for an L-level clique. Obviously, much more work is needed to confirm our intuitions. However, our results suggest that `LOCAL&AND` is the rule for which the multiplex topology cannot be reduced to a monoplex, and—also for this reason—is probably the most appropriate extension of the q-voter model to multiplex networks.

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